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ELECTROMAGNETIC PLANE WAVES IN ANISOTROPIC MEDIA: AN APPROACH USING BIVECTORS

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CONTENTS

The propagation of time-harmonic electromagnetic plane waves in non-absorbing, non-optically active, electrically and magnetically anisotropic media is considered. Both homogeneous and inhomogeneous plane waves are considered. All such solutions to Maxwell equations are obtained for crystals with arbitrary uniform electrical permittivity and magnetic permeability tensors. The addition of the magnetic anisotropy to the electrical anisotropy introduces qualitative changes. For example, for homogeneous linearly polarized waves in magnetically isotropic media the electric displacement vector **D** and the magnetic induction vector **B** are always orthogonal, whereas for magnetically anisotropic media these vectors are generally along the common conjugate radii of pairs of ellipses and are only orthogonal in special cases. Also in magnetically isotropic media a homogeneous wave with **D** and **B** both circularly polarized may propagate along an optic axis. However, for magnetically and electrically anisotropic media there is in general no homogeneous wave for which **D** and **B** are both circularly polarized. For inhomogeneous waves there are similar qualitative changes for magnetically anisotropic media. The description of an inhomogeneous plane wave involves two complex vectors, or bivectors: the amplitude and slowness bivectors. By a systematic use of the properties of bivectors and their associated directional ellipses, many of the results obtained are given a direct geometrical interpretation.

0. Notation and terminology

The summation convention applies to repeated lower case latin suffixes. Lower case bold face letters $\boldsymbol{a}, \boldsymbol{b}, \ldots$, represent real vectors. Real unit vectors are denoted by $\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}, \ldots$. Throughout the paper the unit vectors $\hat{\boldsymbol{m}}, \hat{\boldsymbol{n}}$ are orthogonal. Upper case bold face letters, $\boldsymbol{A}, \boldsymbol{B}, \ldots$, represent bivectors, that is complex vectors. The superscripts $+$ and $-$ refer to the real and imaginary parts: $\boldsymbol{A} = \boldsymbol{A}^{+} + \mathrm{i}\boldsymbol{A}^{-}$, $\boldsymbol{B} = \boldsymbol{B}^{+} + \mathrm{i}\boldsymbol{B}^{-}, \ldots$. A bar denotes the complex conjugate: $\bar{\boldsymbol{A}} = \boldsymbol{A}^{+} - \mathrm{i}\boldsymbol{A}^{-}$, $\bar{\boldsymbol{B}} = \boldsymbol{B}^{+} - \mathrm{i}\boldsymbol{B}^{-}$. Throughout the paper $\boldsymbol{C}$ denotes the bivector $\boldsymbol{C} = m\hat{\boldsymbol{m}} + \mathrm{i}\hat{\boldsymbol{n}}$, where m is a real number. Two ellipses are said to be similar if they have the same aspect ratio (ratio of major to minor axis). Two ellipses are said to be similarly situated if they are concentric, coplanar and their major axes are in the same direction.

A section by a plane through the centre of an ellipsoid is said to be a 'central elliptical section' or for brevity an 'elliptical section'. Two ellipsoids are said to be similar and similarly situated if, being concentric, every pair of central elliptical sections of the two ellipsoids is a pair of similar and similarly situated ellipses.

1. Introduction

We consider the propagation of time-harmonic electromagnetic plane waves in homogeneous electrically and magnetically anisotropic media which are non-absorbing and non-optically active.

Both homogeneous and non homogeneous plane waves are considered. Inhomogeneous plane waves, sometimes called 'evanescent waves' are those for which the planes of constant amplitude are not the same as the planes of constant phase. They are of importance because they are the basic building blocks that may be combined to form solutions that satisfy initial and boundary conditions. Not only are they of mathematical interest but as Bryngdahl (1973) has pointed out 'their peculiar behaviour can be advantageously used for optical imaging purposes'. Among the areas in which they may be used, Bryngdahl cites the investigation of surface topography, film thickness, refractive index and also frequency conversion and holography.

The study of homogeneous waves in magnetically isotropic but electrically anisotropic media has a long history (see, for example, Born and Wolf 1980). Recently Hayes (1987) presented a systematic investigation of inhomogeneous electromagnetic waves in such crystals, based upon a method he had previously proposed (Hayes 1984).

Here we consider both electrical and magnetic anisotropy: the electrical permittivity κ and the magnetic permeability μ are assumed to be arbitrary real positive definite symmetric second-order tensors. There are many materials that exhibit magnetic anisotropy (see, for example, Morrish 1965; Landolt-Börnstein 1986, 1988). Most classical studies of electromagnetic waves, especially in view of applications to optics, are restricted to the case of magnetically isotropic media. But there are indeed cases when it is reasonable to take both electrical and magnetic anisotropy into account when dealing with wave propagation. We present some examples.

Even though ferrites (ferrimagnetic materials) at microwave frequencies (1 to 100 GHz) are generally assumed to be magnetically anisotropic but electrically isotropic (see, for example, Waldron 1970), the development of new materials and technologies for microwave applications has led several authors to consider media for which κ and μ are both anisotropic (Graglia & Uslenghi 1987; Morgan *et al.* 1987; Monzon 1988).

In the study of liquid crystals of the nematic type, the electrical permittivity κ and the magnetic permeability μ are taken to be isotropic tensor functions of a single vector, the director $\hat{\boldsymbol{d}}$. The tensors κ and μ have the form (Ericksen 1962; Leslie 1987) $\kappa = \kappa_0 1 + \kappa_a \hat{\boldsymbol{d}} \otimes \hat{\boldsymbol{d}}$, $\mu = \mu_0 1 + \mu_a \hat{\boldsymbol{d}} \otimes \hat{\boldsymbol{d}}$, where κ_0, κ_a, μ_0, μ_a are constants. According to Leslie (1987), μ_a, 'the diamagnetic anisotropy' is generally positive for most nematics, while κ_a, 'the dielectric anisotropy' may be either positive or negative, depending on the nematic under consideration. Thus for nematic liquid crystals κ and μ are both anisotropic, but have the same principal axes

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so that the corresponding electromagnetic constitutive equations are a special case of those considered in this paper.

Similarly, when studying the photoelastic effect in finitely deformed homogeneous isotropic elastic materials, both the electrical permittivity κ and the magnetic permeability μ may be assumed to be isotropic tensor functions of the finite strain tensor (Smith & Rivlin 1970). Then κ and μ are both anisotropic but have also the same principal axes.

Also when studying electro- and magneto-optical effects in non dissipative polarizable and magnetizable isotropic dielectrics (Boulanger *et al.* 1973; Toupin & Rivlin 1961), electrically and magnetically anisotropic behaviours arise together as a consequence of an imposed strong static electric or magnetic field.

High-temperature superconductors (such as $YBa_2Cu_3O_{7-x}$) are known to have highly anisotropic electrical and magnetic properties (Markiewicz *et al.* 1988; Dinger *et al.* 1987; Crabtree *et al.* 1987). When considering optical properties of such materials it is important to have available a theory that takes into account both the electric and magnetic anisotropy.

Finally, although crystals that are both electrically and magnetically anisotropic in their natural state are uncommon, we note that $BaMnF_4$ and $Cu(HCOO)_2.4H_2O$ (cupric formate tetra-hydrate) exhibit both electrical and magnetic anisotropic properties (Landolt-Börnstein 1987).

In the present paper we obtain all homogeneous and inhomogeneous wave solutions to Maxwell's equations for arbitrary uniform real symmetric electrical permittivity and magnetic permeability tensors.

Throughout, in describing the waves we use complex vectors or bivectors (to use the word of Hamilton (1853) and Gibbs (1881)). The link between bivectors and inhomogeneous plane waves is that the field, for example the electric field $\mathbf{E}$, corresponding to a train of such waves is described in terms of two bivectors. Thus, $\mathbf{E} = [\boldsymbol{E} \exp \mathrm{i}\omega(\boldsymbol{S}\cdot\boldsymbol{x}-t)]^{+}$. Here the frequency ω is real, the amplitude bivector $\boldsymbol{S}$ determines the ellipse of polarization and the slowness bivector $\boldsymbol{S}$ determines the planes of constant phase, $\boldsymbol{S}^{+}\cdot\boldsymbol{x} = \text{const.}$, the planes of constant amplitude, $\boldsymbol{S}^{-}\cdot\boldsymbol{x} = \text{const.}$, the phase slowness $|\boldsymbol{S}^{+}|$ and the attenuation factor $|\boldsymbol{S}^{-}|$. For non attenuated homogeneous waves the field has the form $\mathbf{E} = [\boldsymbol{E} \exp \mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)]^{+}$ where $\boldsymbol{k}$ is the real wave vector and the phase speed is $\omega/|\boldsymbol{k}|$. Typically, for homogeneous waves, the direction of $\boldsymbol{k}$ is chosen, and an eigenvalue problem is solved to determine the corresponding phase speeds and the corresponding amplitude $\boldsymbol{E}$. For inhomogeneous waves, on the other hand, the directions of the normals to the planes of constant phase and to the planes of constant amplitude may not be chosen arbitrarily. Instead, as suggested by Hayes (1984), we write $\boldsymbol{S} = N\boldsymbol{C}$, where $\boldsymbol{C} = m\hat{\boldsymbol{m}} + \mathrm{i}\hat{\boldsymbol{n}}$, with $\hat{\boldsymbol{m}}\cdot\hat{\boldsymbol{n}} = 0$, $|\hat{\boldsymbol{m}}| = |\hat{\boldsymbol{n}}| = 1$, and m is a real scalar. If $\boldsymbol{C}$ is prescribed, then N and the corresponding amplitude bivector $\boldsymbol{E}$ may be determined from an eigenvalue problem. Prescription of $\boldsymbol{C}$ is equivalent to the prescription of an ellipse – the directional ellipse, to use Gibb's phrase – whose principal semi axes are of length $|m|$ and 1, and lie along $\hat{\boldsymbol{m}}$ and $\hat{\boldsymbol{n}}$ respectively. Thus, whereas a direction is prescribed in the case of homogeneous waves, a directional ellipse is prescribed in the case of inhomogeneous waves. For given $\boldsymbol{C}$, if N is known, then the directions of the normals to the planes of constant phase and the normals to the planes of constant amplitude are determined. They lie along a pair of conjugate directions of the directional ellipse (Hayes 1984).

The medium is assumed to be such that the magnetic induction field $\mathbf{B}$ is linearly related to the magnetic intensity field $\mathbf{H}$ through $\mathbf{B} = \mu\mathbf{H}$ where μ is the magnetic permeability tensor. Also, the electric displacement field $\mathbf{D}$ is linearly related to the electric intensity field through

$\mathbf{D} = \kappa\mathbf{E}$ where κ is the electric permittivity tensor. In dealing with inhomogeneous plane waves all the field quantities are assumed to have the form $(\mathbf{B}, \mathbf{D}, \mathbf{E}, \mathbf{H}) = [(\boldsymbol{B}, \boldsymbol{D}, \boldsymbol{E}, \boldsymbol{H}) \exp \mathrm{i}\omega(\boldsymbol{S}\cdot\boldsymbol{x} - t)]^+$, where the amplitude bivectors $\boldsymbol{B}, \boldsymbol{D}, \boldsymbol{E}, \boldsymbol{H}$ are assumed constant.

Properties of magnetically isotropic crystals are described in terms of the κ-metric ellipsoid (Fresnel ellipsoid). For the analytical treatment it is then convenient to introduce cartesian axes along the principal directions of κ. The crystals are classified according to the number of central circular sections of the Fresnel ellipsoid (biaxial if there are two central circular sections, uniaxial if there is one and isotropic if all central sections are circular). Here, however, where the electrical and magnetic anisotropies are placed on an equal footing, the properties of the crystals are described in terms of two ellipsoids, one corresponding to the magnetic permeability μ and the other corresponding to the electrical permittivity κ. We then introduce a set of oblique axes that diagonalize simultaneously κ and μ, that is axes which are conjugate together with respect to both ellipsoids. These oblique axes play the same rôle as the principal axes of κ in the study of magnetically isotropic media. The classification of the crystals is given in terms of the two ellipsoids (chosen concentric without loss in generality), the basis of the classification being that, in general, for any pair of ellipsoids there is at least one central plane that cuts the two ellipsoids in a pair of similar and similarly situated ellipses. If there are two such central planes the crystal is said to be biaxial, if there is only one such central plane the crystal is said to be uniaxial, and if every central plane cuts the two ellipsoids in pairs of similar and similarly situated ellipses the crystal is said to be pseudo-isotropic. For a pseudo-isotropic crystal, the two ellipsoids are thus similar and similarly situated. The adjective 'pseudo' is used, for the crystal need not be isotropic: the wave speed will depend upon the direction of propagation. The classification we have introduced is consistent with the standard classification of magnetically isotropic crystals, for in this case the μ-metric ellipsoid is a sphere.

In §§2 and 3, we derive from the basic equations (§2) a propagation condition (§3) that may be formulated either as an eigenvalue problem for the amplitude $\boldsymbol{E}$ of the electric field or as an eigenvalue problem for the amplitude $\boldsymbol{H}$ of the magnetic field.

From the secular equation and propagation condition we derive (§4) orthogonality relations for one wave solution. Many of these orthogonality relations are of the form $\boldsymbol{P}^{\mathrm{T}}g\boldsymbol{Q} = 0$ where g may be μ, μ^{-1}, κ or κ^{-1} and $\boldsymbol{P}$ or $\boldsymbol{Q}$ may be $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}$ or $\boldsymbol{B}$. These relations are interpreted (Appendix B) through the introduction of the concept of orthogonal projection with respect to the metric g or g-projection (the g-projection onto a plane α being the parallel projection along the direction conjugate to the plane α with respect to the g-metric ellipsoid).

For given $\boldsymbol{C}$ there are in general two different non-zero roots of the secular equation. There are corresponding orthogonality relations involving both fields. These relations are derived and interpreted (§4.2). Typically, if $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ are the amplitude bivectors for the two waves, then $\boldsymbol{E}_1^{\mathrm{T}}\kappa\boldsymbol{E}_2 = 0$, which means that the κ-projection of the ellipse of $\boldsymbol{E}_2$ upon the plane of $\boldsymbol{E}_1$ is similar and similarly situated to the polar reciprocal of the ellipse of $\boldsymbol{E}_1$ with respect to the elliptical section of the κ-metric ellipsoid by the plane of the ellipse of $\boldsymbol{E}_1$.

For certain choices of the directional ellipse the secular equation may have double non-zero roots. Orthogonality relations for waves corresponding to these double roots are presented and interpreted (§4.3). The interpretation is facilitated by generalizing the usual concept of isotropy for a bivector $\boldsymbol{A}$, namely $\boldsymbol{A}\cdot\boldsymbol{A} = 0$, to that of isotropy with respect to a positive definite symmetric matrix g(say), that is $\boldsymbol{A}^{\mathrm{T}}g\boldsymbol{A} = 0$. It is proved (Appendix C) that a necessary and sufficient condition that the complex symmetric matrix X have an isotropic eigenbivector $\boldsymbol{A}$ with respect to the metric g (that is, $X\boldsymbol{A} = \lambda g\boldsymbol{A}$, with $\boldsymbol{A}^{\mathrm{T}}g\boldsymbol{A} = 0$) is that X have a double or triple eigenvalue with respect to the metric g (that is the equation $\det|X-\lambda g| = 0$ has a double or triple root for λ).

General results are derived (§5.1) relating the mean energy flux (Poynting) vector and the mean energy density for inhomogeneous waves. The form of the mean energy flux vector is obtained for the combined motion of two waves propagating with the same slowness (§5.2).

Next (§6) we obtain universal relations involving the base speeds and attenuation factors. These relations are valid independently of the choice of the constitutive tensors κ and μ.

Homogeneous plane waves corresponding to non-repeated eigenvalues are next considered (§7). The waves are linearly polarized. For such waves propagating in the direction $\hat{\boldsymbol{n}}$, the amplitude vectors $\boldsymbol{B}$ and $\boldsymbol{D}$ must lie along the common conjugate directions of the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane $\Pi(\hat{\boldsymbol{n}})$ whose normal is $\hat{\boldsymbol{n}}$. (Every pair of coplanar concentric ellipses possess one, and only one, pair of common conjugate directions unless these ellipses are similar and similarly situated). If $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are unit vectors along the common conjugate directions, it is seen that two waves may propagate in a given direction $\hat{\boldsymbol{n}}$, one with **D** field amplitude along $\boldsymbol{d}_1$ (and **B** field amplitude along $\boldsymbol{d}_2$) and the other with **D** field amplitude along $\boldsymbol{d}_2$ (and **B** field amplitude also $\boldsymbol{d}_1$). Of course, the polarization states of these two waves are in general not orthogonal, as is the case for magnetically isotropic crystals (or electrically isotropic crystals). The slowness N_1 (say) of the wave propagating along $\hat{\boldsymbol{n}}$ with **D** along $\boldsymbol{d}_1$ is equal to the area of the parallelogram formed by the radius along $\boldsymbol{d}_1$ to the κ^{-1}-metric ellipsoid and the radius along $\boldsymbol{d}_2$ to the μ^{-1}-ellipsoid. Similarly the slowness N_2 (say) of the wave propagating along $\hat{\boldsymbol{n}}$ with **D** along $\boldsymbol{d}_2$ is equal to the area of the parallelogram formed by the radius along $\boldsymbol{d}_2$ to the μ^{-1}-metric ellipsoid and the radius along $\boldsymbol{d}_1$ to the μ^{-1}-metric ellipsoid. This is the generalization of the classical result (e.g. Born & Wolf 1980, p. 673) for magnetically isotropic crystals ($\mu = \mu 1$) in which the **D** field amplitudes of the two waves are along the principal axes of the elliptical section of the κ^{-1}-metric ellipsoid by the plane $\Pi(\hat{\boldsymbol{n}})$ and the slownesses of the two waves are equal to $\sqrt{\mu}$ times the lengths of the principal semi axes of this elliptical section.

In §8, circularly polarized *homogeneous* plane waves are considered. For magnetically isotropic crystals it was shown by Hayes (1984) that circularly polarized fields are possible for homogeneous waves if and only if the secular equation has a double root. It is shown here that if for a given $\hat{\boldsymbol{n}}$ (the direction of phase propagation) the secular equation has a double root, then the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane $\Pi(\hat{\boldsymbol{n}})$ orthogonal to $\hat{\boldsymbol{n}}$ are similar and similarly situated, and conversely, if the elliptical sections by the plane $\Pi(\hat{\boldsymbol{n}})$ are similar and similarly situated then the secular equation has a double root. The corresponding directions $\hat{\boldsymbol{n}}$ are called 'generalized optic axes'. For given κ and μ there is either one such direction (uniaxial case), two such (biaxial) or an infinity (pseudo-isotropic case). These directions are obtained using the set of oblique axes that diagonalize simultaneously κ and μ. For $\hat{\boldsymbol{n}}$ along a generalized optic axis, $\boldsymbol{D}$ (or $\boldsymbol{B}$) may in particular be chosen to be isotropic in the plane $\Pi(\hat{\boldsymbol{n}})$, and then the ellipse of $\boldsymbol{B}$ (or $\boldsymbol{D}$) is the polar reciprocal of the circle of $\boldsymbol{D}$ (or $\boldsymbol{B}$) with respect to either of the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane $\Pi(\hat{\boldsymbol{n}})$. Then the **D** (or **B**) field is circularly polarized, whereas the **B** (or **D**) field is elliptically polarized. Also, for $\hat{\boldsymbol{n}}$ along a generalized optic axis, the bivectors $\boldsymbol{D}$ and $\boldsymbol{B}$ may be chosen to be parallel so that their ellipses are similar and similarly situated. In this case, the fields **D** and **B** are at any time along conjugate directions of these ellipses. This pair of conjugate directions 'rotates' (but not rigidly) with time. This generalizes the usual properties of circularly polarized homogeneous waves in magnetically isotropic crystals.

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For a biaxial crystal the directions of the generalized optic axes are the only two directions in which homogeneous waves with $\boldsymbol{D}$ and $\boldsymbol{B}$ parallel may propagate. For a uniaxial crystal there is one generalized optic axis; it too is the only direction in which homogeneous waves with $\boldsymbol{D}$ and $\boldsymbol{B}$ parallel may propagate. For pseudo-isotropic crystals every direction is a generalized optic axis, and homogeneous waves with $\boldsymbol{D}$ and $\boldsymbol{B}$ parallel may propagate in every direction, as is the case for isotropic media. However, for pseudo-isotropic crystals the phase speed (or the refractive index) of these waves in general depends upon the direction of propagation.

Inhomogeneous plane wave solutions for biaxial crystals are considered in §9. Explicit forms of the secular equation and of the propagation condition are written down (§9.1) using the set of oblique axes that diagonalize simultaneously κ and μ. Of course one root of the secular equation for N^{-2} is always zero, but for certain choices of $\boldsymbol{C}$ both of the two other roots are also zero so that no wave propagation is possible (§9.2). The existence of such zero roots is hardly surprising when the ordinary wave equation $\nabla^2\phi = \partial^2\phi/\partial t^2$ is considered. For, letting $\phi = \alpha \exp \mathrm{i}\omega(N\boldsymbol{C}\cdot\boldsymbol{x}-t)$, we have $N^2\boldsymbol{C}\cdot\boldsymbol{C} = 1$. Taking $\boldsymbol{C} = \hat{\boldsymbol{m}}+\mathrm{i}\hat{\boldsymbol{n}}$, so that $\boldsymbol{C}$ is isotropic ($\boldsymbol{C}\cdot\boldsymbol{C} = 0$), we note that there is no progressive plane wave solution of the wave equation with an isotropic slowness bivector $\boldsymbol{S}$. The choice of an isotropic $\boldsymbol{C}$ has to be rejected. However, we note that $\phi = \alpha \exp \mathrm{i}\omega\boldsymbol{C}\cdot\boldsymbol{x}$, with $\boldsymbol{C}\cdot\boldsymbol{C} = 0$, is a time-independent solution of the wave equation, that is a solution of the Laplace equation $\nabla^2\phi = 0$. The directional ellipses corresponding to the choices of $\boldsymbol{C}$ such that no progressive wave propagation is possible are here similar and similarly situated to certain sections of the κ or μ-metric ellipsoid. We call them 'critical sections': no wave propagation is possible with a directional ellipse similar and similarly situated to a critical section.

The condition for a double (non-zero) root of the secular equation leads to an equation that has simple factors. It is seen (§9.3) that corresponding to the double root there is a wave for which the bivectors $\boldsymbol{D}$ and $\boldsymbol{B}$ are parallel – that is the ellipses of **D** and **B** are coplanar – and are similar and similarly situated.

The possibility of inhomogeneous waves with **E** and **D**, or **B** and **H**, linearly polarized is next considered (§9.4). It is seen that these waves may propagate for certain choices of $\boldsymbol{C}$.

For uniaxial crystals the secular equation has simple factors. In the case of homogeneous waves the roots for N^{-2} correspond to ellipsoidal slowness surfaces (§10.1). For *inhomogeneous* waves, critical sections of the κ and μ-metric ellipsoid are obtained (§10.2). Also, it is seen (§10.3) that corresponding to a double root of the secular equation, the ellipses of **D** and **B** are both similar and similarly situated to either of the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane orthogonal to the generalized optic axis.

For pseudo-isotropic crystals the electrical permittivity tensor κ is a scalar multiple of the magnetic permeability tensor μ. In this case the secular equation has a double root N^{-2} for every bivector $\boldsymbol{C}$. For homogeneous waves this means the two slowness surfaces associated with an uniaxial crystal coalesce into a single ellipsoid. It is seen (§11.1) that there is a double infinity of eigenbivectors $\boldsymbol{E}$ or $\boldsymbol{H}$ corresponding to the double root of the secular equation. Also, in this case, every section of the κ or μ-metric ellipsoid turns out to be a critical section (§11.2). It is also seen (§11.3) that for every choice of $\boldsymbol{C}$ that is not linear, there is an inhomogeneous wave with **E** and **D** linearly polarized and another with **H** and **B** linearly polarized. These two waves are combined (§11.4) to form the general inhomogeneous wave solution. Further (§11.5) for every choice of $\boldsymbol{C}$ there are two waves for which the bivectors $\boldsymbol{D}$ and $\boldsymbol{B}$ are parallel (the ellipses of $\boldsymbol{D}$ and $\boldsymbol{B}$ are both similar, and similarly situated, to a section of κ^{-1} or μ^{-1}-metric ellipsoids).

2. Basic equations

In this section, time harmonic plane waves solutions of Maxwell's equations for an electrically and magnetically anisotropic material are introduced. Both homogeneous and inhomogeneous plane waves are considered.

Maxwell's equations, in the absence of current and charges, are

$$\nabla \cdot \mathbf{D} = 0, \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.2}$$

$$\nabla \times \mathbf{E} + \partial \mathbf{B}/\partial t = 0, \tag{2.3}$$

$$\nabla \times \mathbf{H} - \partial \mathbf{D}/\partial t = 0, \tag{2.4}$$

where $\mathbf{E}$ is the electric intensity, $\mathbf{B}$ the magnetic induction, $\mathbf{D}$ the electric displacement and $\mathbf{H}$ the magnetic intensity.

The constitutive equations for the crystal are taken to be

$$\left.\begin{aligned} \mathbf{D} = \kappa \mathbf{E}, \quad \mathsf{D}_i = \kappa_{ij}\,\mathsf{E}_j, \\ \mathbf{B} = \mu \mathbf{H}, \quad \mathsf{B}_i = \mu_{ij}\,\mathsf{H}_j. \end{aligned}\right\} \tag{2.5}$$

Here κ_{ij} and μ_{ij} are respectively the electric permittivity and magnetic permeability tensors for the medium. They are assumed to be constant, real, symmetric positive definite tensors. In terms of the vacuum permittivity, κ_0, and permeability, μ_0, we have

$$\kappa_{ij} = \kappa_0 K_{ij}, \quad \mu_{ij} = \mu_0 M_{ij}, \tag{2.6}$$

where K and M are respectively the relative permittivity and permeability tensors.

It is assumed that the fields are due to the propagation of an infinite train of inhomogeneous plane waves in the crystal. Thus

$$(\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}) = (\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}, \boldsymbol{B}) \exp \mathrm{i}\omega(\boldsymbol{S}\cdot\boldsymbol{x} - t), \tag{2.7}$$

where the bivectors (complex vectors) $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{D}$, $\boldsymbol{B}$ are independent of position and time, ω is the angular frequency of the waves (real) and $\boldsymbol{x}$ is the position vector of a generic point of space. The bivector $\boldsymbol{S}$ is called the 'slowness' bivector and may be written

$$\boldsymbol{S} = \boldsymbol{S}^{+} + \mathrm{i}\boldsymbol{S}^{-}, \tag{2.8}$$

so that $\boldsymbol{S}^{+}\cdot\boldsymbol{x} = \text{const.}$ are the planes of constant phase and $\boldsymbol{S}^{-}\cdot\boldsymbol{x} = \text{const.}$ are the planes of constant amplitude. The phase speed is $1/|\boldsymbol{S}^{+}|$, and the attenuation factor is $|\boldsymbol{S}^{-}|$.

The electric displacement $\mathbf{D}$ (magnetic induction $\mathbf{B}$) represents an infinite train of elliptically polarized waves, the plane of polarization being that of the directional ellipse (Hayes 1987) of $\boldsymbol{D}$ ($\boldsymbol{B}$).

The electric displacement $\mathbf{D}$ is linearly polarized if

$$\bar{\boldsymbol{D}} \times \boldsymbol{D} = 0, \tag{2.9}$$

where $\bar{\boldsymbol{D}} = \boldsymbol{D}^{+} - \mathrm{i}\boldsymbol{D}^{-}$ is the complex conjugate of $\boldsymbol{D}$. In this case the vectors $\boldsymbol{D}^{+}$ and $\boldsymbol{D}^{-}$ have the same direction and the bivector $\boldsymbol{D}$ is said to be 'linear'. By changing the time origin in (2.7) $\boldsymbol{D}$ may then be taken to be real.

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The electric displacement **D** is circularly polarized if

$$\boldsymbol{D}\cdot\boldsymbol{D} = 0. \tag{2.10}$$

In this case the bivector $\boldsymbol{D}$ is said to be 'isotropic' or 'null'. Similarly, the magnetic induction **B** is linearly polarized if $\bar{\boldsymbol{B}}\times\boldsymbol{B} = 0$ ($\boldsymbol{B}$ is a linear bivector) and circularly polarized if $\boldsymbol{B}\cdot\boldsymbol{B} = 0$ ($\boldsymbol{B}$ is an isotropic bivector).

The slowness bivector $\boldsymbol{S}$ may also be written (Hayes 1984)

$$\boldsymbol{S} = N\boldsymbol{C}, \tag{2.11}$$

where

$$N = T\,\mathrm{e}^{\mathrm{i}\phi}, \quad \boldsymbol{C} = m\hat{\boldsymbol{m}} + \mathrm{i}\hat{\boldsymbol{n}}. \tag{2.12}$$

Here N is a complex number with modulus T and argument ϕ, $\hat{\boldsymbol{m}}$ and $\hat{\boldsymbol{n}}$ are orthogonal unit vectors, and m is a real number. To determine all the possible slowness bivectors, $\boldsymbol{S}$, the complex number N has to be found for every choice of the bivector $\boldsymbol{C}$, that is for every choice of the orthogonal unit vectors $\hat{\boldsymbol{m}}$, $\hat{\boldsymbol{n}}$ and of the real number m.

Associated with $\boldsymbol{C}$ is an ellipse, its directional ellipse, which has one semi-axis of unit length, the other being of length $|m|$. Then, when N has been determined for a choice of such an ellipse, $\boldsymbol{S}$ is known, and $\boldsymbol{S}^+$ and $\boldsymbol{S}^-$ are conjugate semi-diameters of an ellipse similar and similarly situated with respect to the ellipse of $\boldsymbol{C}$ (details may be found in Hayes (1984)).

For homogeneous waves $\boldsymbol{S}$ is a linear bivector, and we may assume that the bivector $\boldsymbol{C}$ is a real unit vector $\hat{\boldsymbol{n}}$ (unit vector in the propagation direction):

$$\boldsymbol{S} = N\boldsymbol{C} \quad \text{where} \quad \boldsymbol{C} = \hat{\boldsymbol{n}}. \tag{2.13}$$

The number $c = N^{-1}$ is then the phase speed of the wave (It will be shown in §7 that N^{-1} is always real for homogeneous waves), and if c_0 denotes the speed of light in vacuum ($c_0^{-2} = \kappa_0\mu_0$), c_0/c is the refractive index of the wave.

It is convenient to introduce the dual skew-symmetric tensor Γ associated with the bivector $\boldsymbol{C}$:

$$\Gamma_{ij} = -\epsilon_{ijk}C_k = -\Gamma_{ji}, \quad C_i = -\tfrac{1}{2}\epsilon_{ijk}\Gamma_{jk}, \tag{2.14}$$

where ϵ_{ijk} is the alternating symbol. For any bivector $\boldsymbol{P}$, one has

$$\Gamma\boldsymbol{P} = \boldsymbol{C}\times\boldsymbol{P}, \quad \Gamma_{ij}P_j = (\boldsymbol{C}\times\boldsymbol{P})_i. \tag{2.15}$$

3. The propagation condition

Inserting the expressions (2.7) into Maxwell's equations (2.1)–(2.4), and using (2.11) and (2.15) gives

$$\boldsymbol{C}\cdot\boldsymbol{D} = 0, \quad \boldsymbol{C}\cdot\boldsymbol{B} = 0, \tag{3.1 a, b}$$

$$N\boldsymbol{C}\times\boldsymbol{E} = \boldsymbol{B} \quad \text{or} \quad N\Gamma\boldsymbol{E} = \boldsymbol{B}, \tag{3.1 c}$$

$$N\boldsymbol{C}\times\boldsymbol{H} = -\boldsymbol{D} \quad \text{or} \quad N\Gamma\boldsymbol{H} = -\boldsymbol{D}. \tag{3.1 d}$$

Also, using the constitutive equations (2.5),

$$\boldsymbol{D} = \kappa\boldsymbol{E}, \quad \boldsymbol{B} = \mu\boldsymbol{H}. \tag{3.2}$$

From the equations (3.2) and (3.1 c, d) we have

$$\kappa\boldsymbol{E} = -N\Gamma\mu^{-1}\boldsymbol{B} = -N^2\Gamma\mu^{-1}\Gamma\boldsymbol{E}, \tag{3.3}$$

and hence the propagation condition in terms of $\boldsymbol{E}$ is

$$(\Gamma\mu^{-1}\Gamma+N^{-2}\kappa)\,\boldsymbol{E}=0. \tag{3.4}$$

Similarly we have

$$\mu\boldsymbol{H}=N\Gamma\kappa^{-1}\boldsymbol{D}=-N^2\Gamma\kappa^{-1}\Gamma\boldsymbol{H}, \tag{3.5}$$

and hence the propagation condition in terms of $\boldsymbol{H}$ is

$$(\Gamma\kappa^{-1}\Gamma+N^{-2}\mu)\,\boldsymbol{H}=0. \tag{3.6}$$

We note in passing the relations

$$\left.\begin{aligned}(\Gamma\kappa^{-1}\Gamma+N^{-2}\mu)\,\boldsymbol{H}&=N\Gamma\kappa^{-1}(\Gamma\mu^{-1}\Gamma+N^{-2}\kappa)\,\boldsymbol{E},\\(\Gamma\mu^{-1}\Gamma+N^{-2}\kappa)\,\boldsymbol{E}&=-N\Gamma\mu^{-1}(\Gamma\kappa^{-1}\Gamma+N^{-2}\mu)\,\boldsymbol{H},\end{aligned}\right\} \tag{3.7}$$

which show that equation (3.4) implies equation (3.6) and vice versa.

Equation (3.4) expresses the fact that $\boldsymbol{E}$ is an eigenbivector of the complex symmetric matrix $\Gamma\mu^{-1}\Gamma$ with respect to the real positive definite symmetric matrix κ, with eigenvalue $-N^{-2}$. Similarly equation (3.6) expresses the fact that $\boldsymbol{H}$ is an eigenbivector of the complex symmetric matrix $\Gamma\kappa^{-1}\Gamma$ with respect to the real positive definite symmetric matrix μ, also with eigenvalue $-N^{-2}$.

We now derive alternative forms of the propagation conditions (3.4) and (3.6). Recalling (2.15), the equations (3.4) and (3.6) may also be written as

$$\boldsymbol{C}\times\{\mu^{-1}(\boldsymbol{C}\times\boldsymbol{E})\}+N^{-2}\kappa\boldsymbol{E}=0, \tag{3.8}$$

and

$$\boldsymbol{C}\times\{\kappa^{-1}(\boldsymbol{C}\times\boldsymbol{H})\}+N^{-2}\mu\boldsymbol{H}=0. \tag{3.9}$$

Using now the identity (A 2) of Appendix A, these equations become

$$(\det\mu)^{-1}\mu\{\boldsymbol{C}(\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{E})-\boldsymbol{E}(\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C})\}+N^{-2}\kappa\boldsymbol{E}=0, \tag{3.10}$$

and

$$(\det\kappa)^{-1}\kappa\{\boldsymbol{C}(\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{H})-\boldsymbol{H}(\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C})\}+N^{-2}\mu\boldsymbol{H}=0. \tag{3.11}$$

So that waves may propagate, the equations (3.4) and (3.6) must have non-trivial solutions. This gives two equivalent forms of the secular equation: either

$$\det(\Gamma\mu^{-1}\Gamma+N^{-2}\kappa)=0, \tag{3.12}$$

or

$$\det(\Gamma\kappa^{-1}\Gamma+N^{-2}\mu)=0. \tag{3.13}$$

That these equations are equivalent may be seen from the relations

$$\begin{aligned}\det(\Gamma\mu^{-1}\Gamma+N^{-2}\kappa)\,(\det\kappa)^{-1}&=\det(\Gamma\mu^{-1}\Gamma\kappa^{-1}+N^{-2}1)\\&=\det(\Gamma\kappa^{-1}\Gamma\mu^{-1}+N^{-2}1)\\&=\det(\Gamma\kappa^{-1}\Gamma+N^{-2}\mu)\,(\det\mu)^{-1},\end{aligned} \tag{3.14}$$

the second line following from the fact that for any two square matrices X and Y, the product XY has the same eigenvalues as the product YX.

The secular equation (3.12) or (3.13) leads to the determination of N^{-2} for given Γ or, equivalently, for given $\boldsymbol{C}$. Because $\det(\Gamma\mu^{-1}\Gamma)=\det(\Gamma\kappa^{-1}\Gamma)=0$, one root of the secular equation for N^{-2} is zero. The other two roots are the solutions of the quadratic equation

$$N^{-4}\det\kappa+N^{-2}[\boldsymbol{C}^{\mathrm{T}}(\kappa\mu^{-1}\kappa)\,\boldsymbol{C}-(\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C})\operatorname{tr}(\kappa\mu^{-1})]+(\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C})(\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C})\,(\det\mu)^{-1}=0, \tag{3.15}$$

or equivalently

$$N^{-4}\det\mu+N^{-2}[\boldsymbol{C}^{\mathrm{T}}(\mu\kappa^{-1}\mu)\,\boldsymbol{C}-(\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C})\,\mathrm{tr}\,(\mu\kappa^{-1})]+(\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C})\,(\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C})\,(\det\kappa)^{-1}=0. \quad (3.16)$$

These forms may be read off from the corresponding forms of the secular equation for homogeneous waves given by Smith & Rivlin (1970), by replacing their unit vector $\boldsymbol{n}$ in the propagation direction by our bivector $\boldsymbol{C}$.

4. Orthogonality relations

In this section we derive results concerning the amplitude bivectors $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{D}$, $\boldsymbol{B}$ from the equations (3.1) for a single inhomogeneous plane wave solution. Further we examine the consequences of the propagation conditions (3.4) and (3.6) and of the equations (3.1) when we have two non-zero different roots or a double non-zero root of the secular equation for N^{-2}. For this we need the idea of orthogonality of a pair of bivectors with respect to a real positive definite symmetric metric. Geometrical interpretation of such orthogonality is derived in Appendix B. In a previous paper (Hayes 1984) it was shown that if two bivectors $\boldsymbol{P}$, $\boldsymbol{Q}$ (say) are orthogonal so that the scalar product $\boldsymbol{P}\cdot\boldsymbol{Q}=0$, then the planes of the ellipses of $\boldsymbol{P}$ and $\boldsymbol{Q}$ may not be orthogonal in general and further the orthogonal projection of one ($\boldsymbol{Q}$ say) upon the plane of the ellipse of $\boldsymbol{P}$ is an ellipse similar and similarly situated with respect to the ellipse of $\boldsymbol{P}$ when rotated through a quadrant. Here, however, the orthogonality of two bivectors $\boldsymbol{P}$, $\boldsymbol{Q}$ is with respect to a real positive definite symmetric metric g and takes the form $\boldsymbol{P}^{\mathrm{T}}g\boldsymbol{Q}=0$. It will be seen that many of the relations derived here are of this form, where g may be either μ, κ or μ^{-1}, κ^{-1} and $\boldsymbol{P}$, $\boldsymbol{Q}$ may be $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{D}$, $\boldsymbol{B}$. The ellipsoid $\boldsymbol{x}^{\mathrm{T}}g\boldsymbol{x}=1$ will be called the 'g-metric ellipsoid'. To interpret the equation $\boldsymbol{P}^{\mathrm{T}}g\boldsymbol{Q}=0$ it is convenient to introduce the concept of orthogonal projection with respect to the metric g or g-projection. The g-projection onto a plane α is the parallel projection along the direction conjugate to the plane α with respect to the g-metric ellipsoid. Details may be found in Appendix B.

4.1. *Orthogonality relations for one wave*

As in Hayes (1987) we first note from equation (3.1 a, b) that the planes of the ellipses of $\boldsymbol{D}$ and $\boldsymbol{C}$, and of $\boldsymbol{B}$ and $\boldsymbol{C}$ may not be orthogonal, in general. Further the projection of the ellipses of $\boldsymbol{D}$ and $\boldsymbol{B}$ upon the plane of the ellipse of $\boldsymbol{C}$ are similar and similarly situated. Also these projected ellipses are similar and similarly situated with respect to the ellipse of $\boldsymbol{C}$ when rotated through a quadrant.

From equations (3.1 c, d) we obtain

$$\boldsymbol{E}\cdot\boldsymbol{B}=0, \quad \boldsymbol{D}\cdot\boldsymbol{H}=0, \qquad (4.1\,a, b)$$

and hence, using the constitutive equations (3.2),

$$\boldsymbol{D}^{\mathrm{T}}\kappa^{-1}\boldsymbol{B}=0, \quad \boldsymbol{D}^{\mathrm{T}}\mu^{-1}\boldsymbol{B}=0, \qquad (4.2\,a, b)$$

i.e.

$$D_i\kappa_{ij}^{-1}B_j=0, \quad D_i\mu_{ij}^{-1}B_j=0,$$

and

$$\boldsymbol{E}^{\mathrm{T}}\kappa\boldsymbol{H}=0, \quad \boldsymbol{E}^{\mathrm{T}}\mu\boldsymbol{H}=0, \qquad (4.3\,a, b)$$

i.e.

$$E_i\kappa_{ij}H_j=0, \quad E_i\mu_{ij}H_j=0.$$

From equation (4.1 a) (and (4.1 b)) it follows that the planes of the ellipses of $\boldsymbol{E}$ and $\boldsymbol{B}$ ($\boldsymbol{D}$ and

$\boldsymbol{H}$) may not be orthogonal in general, and that the projection of the ellipse of $\boldsymbol{E}$ ($\boldsymbol{D}$) upon the plane of the ellipse of $\boldsymbol{B}$ ($\boldsymbol{H}$) is similar and similarly situated with respect to the ellipse of $\boldsymbol{B}$ ($\boldsymbol{H}$) when rotated through a quadrant.

Using the results of the Appendix B (property 4) it follows from equation (4.2*a*) (and (4.2*b*)) that the κ^{-1} (μ^{-1}) projection of the ellipse of $\boldsymbol{B}$ upon the plane of the ellipse of $\boldsymbol{D}$ is similar, and similarly situated, to the polar reciprocal of the ellipse of $\boldsymbol{D}$ with respect to the elliptical section of the κ^{-1} (μ^{-1})-metric ellipsoid by the plane of the bivector $\boldsymbol{D}$. In an analogous way, it follows from equation (4.3*a*) (and (4.3*b*)) that the κ (μ)-projection of the ellipse of $\boldsymbol{H}$ upon the plane of the ellipse of $\boldsymbol{E}$ is similar, and similarly situated, to the polar reciprocal of the ellipse of $\boldsymbol{E}$ with respect to the elliptical section of the κ (μ)-metric ellipsoid by the plane of the bivector $\boldsymbol{E}$.

Taking the dot product of (3.1*c*) with $\boldsymbol{H}$ and the dot product of (3.1*d*) with $\boldsymbol{E}$, and adding, we obtain

$$\boldsymbol{D}\cdot\boldsymbol{E} = \boldsymbol{B}\cdot\boldsymbol{H} = \sigma, \tag{4.4}$$

and hence, using the constitutive equations (3.2),

$$\boldsymbol{E}^{\mathrm{T}}\kappa\boldsymbol{E} = \boldsymbol{H}^{\mathrm{T}}\mu\boldsymbol{H} = \boldsymbol{D}^{\mathrm{T}}\kappa^{-1}\boldsymbol{D} = \mathrm{B}^{\mathrm{T}}\mu^{-1}\boldsymbol{B} = \sigma. \tag{4.5}$$

Moreover, using Maxwell's equations (3.1*a*, *d*) we obtain

$$\boldsymbol{D}\times\boldsymbol{B} = N\boldsymbol{D}\times(\boldsymbol{C}\times\boldsymbol{E}) = N(\boldsymbol{D}\cdot\boldsymbol{E})\,\boldsymbol{C} = N\sigma\boldsymbol{C}; \tag{4.6}$$

with σ given by (4.5).

Also, from (4.3) we deduce that if $\kappa\boldsymbol{E}$ and $\mu\boldsymbol{E}$ are not parallel, then $\boldsymbol{H}$ must be of the form

$$\boldsymbol{H} = \beta(\kappa\boldsymbol{E})\times(\mu\boldsymbol{E}), \tag{4.7}$$

for some scalar β. Inserting this into (4.5) and using the identity (A 3) of Appendix A, we find

$$\boldsymbol{E}^{\mathrm{T}}\kappa\boldsymbol{E} = \beta^2\,(\det\mu)\,[(\boldsymbol{E}^{\mathrm{T}}\kappa\mu^{-1}\kappa\boldsymbol{E})\,(\boldsymbol{E}^{\mathrm{T}}\mu\boldsymbol{E}) - (\boldsymbol{E}^{\mathrm{T}}\kappa\boldsymbol{E})^2], \tag{4.8}$$

and hence, in general, $\boldsymbol{H}$ may be obtained from

$$(\det\mu)^{\frac{1}{2}}[(\boldsymbol{E}^{\mathrm{T}}\kappa\mu^{-1}\kappa\boldsymbol{E})\,(\boldsymbol{E}^{\mathrm{T}}\mu\boldsymbol{E}) - (\boldsymbol{E}^{\mathrm{T}}\kappa\boldsymbol{E})^2]^{\frac{1}{2}}\boldsymbol{H} = \pm(\boldsymbol{E}^{\mathrm{T}}\kappa\boldsymbol{E})^{\frac{1}{2}}(\kappa\boldsymbol{E})\times(\mu\boldsymbol{E}). \tag{4.9}$$

Thus $\boldsymbol{H}$ is determined to within a sign by $\boldsymbol{E}$. Equivalently, because $\boldsymbol{E} = \kappa^{-1}\boldsymbol{D}$, $\boldsymbol{H}$ is also determined to within a sign by $\boldsymbol{D}$. For given $\boldsymbol{C}$, note from (3.1*c*), that if $\boldsymbol{E}$ is changed to $-\boldsymbol{E}$, then $\boldsymbol{B}$, and consequently $\boldsymbol{H}$, must be changed to $-\boldsymbol{B}$, and $-\boldsymbol{H}$ respectively. Equation (4.9) involves even functions of $\boldsymbol{E}$ and it is for this reason that the ambiguity ($\pm$) in sign in equation (4.9) must be retained. In an analogous way, $\boldsymbol{E}$ may be obtained in terms of $\boldsymbol{H}$ from

$$(\det\kappa)^{\frac{1}{2}}[(\boldsymbol{H}^{\mathrm{T}}\mu\kappa^{-1}\mu\boldsymbol{H})\,(\boldsymbol{H}^{\mathrm{T}}\kappa\boldsymbol{H}) - (\boldsymbol{H}^{\mathrm{T}}\kappa\boldsymbol{H})^2]^{\frac{1}{2}}\boldsymbol{E} = \pm(\boldsymbol{H}^{\mathrm{T}}\mu\boldsymbol{H})^{\frac{1}{2}}(\mu\boldsymbol{H})\times(\kappa\boldsymbol{H}). \tag{4.10}$$

Of course, as $\boldsymbol{H} = \mu^{-1}\boldsymbol{B}$, $\boldsymbol{E}$ may also be obtained in terms of $\boldsymbol{B}$. Similarly the relations between $\boldsymbol{B}$ and $\boldsymbol{D}$ are

$$(\det\mu)^{-\frac{1}{2}}[(\boldsymbol{D}^{\mathrm{T}}\kappa^{-1}\mu\kappa^{-1}\boldsymbol{D})\,(\boldsymbol{D}^{T}\mu^{-1}\boldsymbol{D}) - (\boldsymbol{D}^{\mathrm{T}}\kappa^{-1}\boldsymbol{D})^2]^{\frac{1}{2}}\boldsymbol{B} = \pm(\boldsymbol{D}^{\mathrm{T}}\kappa^{-1}\boldsymbol{D})^{\frac{1}{2}}(\kappa^{-1}\boldsymbol{D})\times(\mu^{-1}\boldsymbol{B}), \tag{4.11}$$

$$(\det\kappa)^{-\frac{1}{2}}[(\boldsymbol{B}^{\mathrm{T}}\mu^{-1}\kappa\mu^{-1}\boldsymbol{B})\,(\boldsymbol{B}^{\mathrm{T}}\kappa^{-1}\boldsymbol{B}) - (\boldsymbol{B}^{\mathrm{T}}\kappa^{-1}\boldsymbol{B})^2]^{\frac{1}{2}}\boldsymbol{D} = \pm(\boldsymbol{B}\mu^{-1}\boldsymbol{B})^{\frac{1}{2}}\,(\mu^{-1}\boldsymbol{B})\times(\kappa^{-1}\boldsymbol{B}). \tag{4.12}$$

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Finally, using the identity (A 3) of Appendix A and the equations (3.1), we obtain

$$\left.\begin{aligned} \boldsymbol{D}^{\mathrm{T}}\mu^{-1}\boldsymbol{D} = N^2(\boldsymbol{C}\times\boldsymbol{H})^{\mathrm{T}}\mu^{-1}(\boldsymbol{C}\times\boldsymbol{H}) &= N^2(\det\mu)^{-1}(\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C})\,(\boldsymbol{H}^{\mathrm{T}}\mu\boldsymbol{H}) \\ &= N^2(\det\mu)^{-1}(\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C})\,(\boldsymbol{D}^{\mathrm{T}}\kappa^{-1}\boldsymbol{D}), \\ \boldsymbol{B}^{\mathrm{T}}\kappa^{-1}\boldsymbol{B} = N^2(\boldsymbol{C}\times\boldsymbol{E})^{\mathrm{T}}\kappa^{-1}(\boldsymbol{C}\times\boldsymbol{E}) &= N^2(\det\kappa)^{-1}(\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C})\,(\boldsymbol{E}^{\mathrm{T}}\kappa\boldsymbol{E}) \\ &= N^2(\det\kappa)^{-1}(\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C})\,(\boldsymbol{B}^{\mathrm{T}}\mu^{-1}\boldsymbol{B}), \end{aligned}\right\} \tag{4.13}$$

and

$$\left.\begin{aligned} \boldsymbol{E}^{\mathrm{T}}\kappa\mu^{-1}\kappa\boldsymbol{E} &= N^2(\det\mu)^{-1}(\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C})\,(\boldsymbol{E}^{\mathrm{T}}\kappa\boldsymbol{E}), \\ \boldsymbol{H}^{\mathrm{T}}\mu\kappa^{-1}\mu\boldsymbol{H} &= N^2(\det\kappa)^{-1}(\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C})\,(\boldsymbol{H}^{\mathrm{T}}\mu\boldsymbol{H}). \end{aligned}\right\} \tag{4.14}$$

4.2. *Orthogonality relations for two waves*

Let N_1^{-2} and N_2^{-2} be two different roots of the secular equation and let $\boldsymbol{E}_1, \boldsymbol{E}_2$ and $\boldsymbol{H}_1, \boldsymbol{H}_2$ be the corresponding eigenbivectors of the symmetric matrices $\Gamma\mu^{-1}\Gamma$ and $\Gamma\kappa^{-1}\Gamma$ with respect to κ and μ respectively. It is easily shown, by the standard argument, that $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ are orthogonal with respect to κ, that is

$$\boldsymbol{E}_1^{\mathrm{T}}\kappa\boldsymbol{E}_2 = 0, \tag{4.15}$$

and that $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ are orthogonal with respect to μ, that is

$$\boldsymbol{H}_1^{\mathrm{T}}\mu\boldsymbol{H}_2 = 0. \tag{4.16}$$

Also, from the constitutive equations (3.2), the corresponding $\boldsymbol{D}$'s and $\boldsymbol{B}$'s satisfy

$$\boldsymbol{D}_1^{\mathrm{T}}\kappa^{-1}\boldsymbol{D}_2 = 0, \tag{4.17}$$

$$\boldsymbol{B}_1^{\mathrm{T}}\mu^{-1}\boldsymbol{B}_2 = 0. \tag{4.18}$$

Using the results of the Appendix B (property 4) it follows from equation (4.15) (and (4.16)) that the κ (μ)-projection of the ellipse of $\boldsymbol{E}_2$ ($\boldsymbol{H}_2$) upon the plane of ellipse of $\boldsymbol{E}_1$ ($\boldsymbol{H}_1$) is similar, and similarly situated, to the polar reciprocal of the ellipse of $\boldsymbol{E}_1$ ($\boldsymbol{H}_1$) with respect to the elliptical section of the κ (μ) metric ellipsoid by the plane of the bivector $\boldsymbol{E}_1$ ($\boldsymbol{H}_1$). Similarly, one may read off conclusions from equation (4.17) (and (4.18)) by replacing $\boldsymbol{E}_1, \boldsymbol{E}_2, \kappa$ ($\boldsymbol{H}_1$, $\boldsymbol{H}_2, \mu$) by $\boldsymbol{D}_1, \boldsymbol{D}_2, \kappa^{-1}$ ($\boldsymbol{B}_1, \boldsymbol{B}_2, \mu^{-1}$) in the previous sentence.

Moreover, using (3.1), we obtain

$$\boldsymbol{D}_1\times\boldsymbol{B}_2 = -N_1(\boldsymbol{C}\times\boldsymbol{H}_1)\times\boldsymbol{B}_2 = N_1(\boldsymbol{H}_1\cdot\boldsymbol{B}_2)\,\boldsymbol{C} = N_1(\boldsymbol{H}_1^{\mathrm{T}}\mu\boldsymbol{H}_2)\,\boldsymbol{C} = 0, \tag{4.19}$$

$$\boldsymbol{D}_2\times\boldsymbol{B}_1 = N_1\boldsymbol{D}_2\times(\boldsymbol{C}\times\boldsymbol{E}_1) = N_1(\boldsymbol{D}_2\cdot\boldsymbol{E}_1)\,\boldsymbol{C} = N_1(\boldsymbol{E}_2^{\mathrm{T}}\kappa\boldsymbol{E}_1)\,\boldsymbol{C} = 0, \tag{4.20}$$

which show that the bivectors $\boldsymbol{D}_1$ and $\boldsymbol{B}_2$ are parallel, as are the bivectors $\boldsymbol{D}_2$ and $\boldsymbol{B}_1$. Hence the ellipse of $\boldsymbol{B}_2$ ($\boldsymbol{B}_1$) is similar, and similarly situated, to the ellipse of $\boldsymbol{D}_1$ ($\boldsymbol{D}_2$).

4.3. *Orthogonality relations for waves corresponding to a double root*

Let N^{-2} be a double root of the secular equation. Thus $(-N^{-2})$ is a double eigenvalue of the eigenvalue problems (3.4) and (3.6). Hayes (1984) has shown that a necessary and sufficient condition for a complex symmetric 3×3 matrix to have an isotropic eigenbivector is that this matrix has a double eigenvalue. It may be shown that this theorem, proved in Hayes (1984) for eigenbivectors and eigenvalues with respect to the unit matrix, remains valid for eigenbivectors and eigenvalues with respect to a real positive definite symmetric matrix, the usual concept of isotropic bivector being replaced by that of isotropic bivector with respect to this matrix (see Appendix C).

Two cases have to be considered for the eigenbivectors $\boldsymbol{E}, \boldsymbol{H}$ corresponding to the double eigenvalue $(-N^{-2})$.

Case 1: *simple infinity of eigenbivectors*

Corresponding to the double root N^{-2}, there is a simple infinity of eigenbivectors $\boldsymbol{E}$ ($\boldsymbol{H}$) of the symmetric matrix $\Gamma\mu^{-1}\Gamma$ ($\Gamma\kappa^{-1}\Gamma$) with respect to κ (μ), and they are isotropic with respect to κ (μ), that is $\boldsymbol{E}^{\mathrm{T}}\kappa\boldsymbol{E} = 0$ ($\boldsymbol{H}^{\mathrm{T}}\mu\boldsymbol{H} = 0$). Thus corresponding to N^{-2} there is one wave solution such that σ, given by (4.5), is zero:

$$\boldsymbol{E}^{\mathrm{T}}\kappa\boldsymbol{E} = \boldsymbol{H}^{\mathrm{T}}\mu\boldsymbol{H} = \boldsymbol{D}^{\mathrm{T}}\kappa^{-1}\boldsymbol{D} = \boldsymbol{B}^{\mathrm{T}}\mu^{-1}\boldsymbol{B} = 0. \tag{4.21}$$

Thus for this wave, $\boldsymbol{D}$ and $\boldsymbol{B}$ are isotropic with respect to κ^{-1} and μ^{-1} respectively, and owing to (4.6) they are parallel:

$$\boldsymbol{D} \times \boldsymbol{B} = 0. \tag{4.22}$$

So, $\boldsymbol{D}$ and $\boldsymbol{B}$ are isotropic together with respect to κ^{-1} and μ^{-1}

In the special cases of magnetically or electrically isotropic crystals ($\mu_{ij} = \mu\delta_{ij}$ or $\kappa_{ij} = \kappa\delta_{ij}$) conditions (4.21) and (4.22) imply that the fields $\mathbf{D}$ and $\mathbf{B}$ are circularly polarized, so that a wave obeying (4.21) may be regarded as a generalization of a circularly polarized wave.

Case 2: *double infinity of eigenbivectors*

Corresponding to the double root N^{-2} there is a double infinity of eigenbivectors $\boldsymbol{E}$ ($\boldsymbol{H}$) of the symmetric matrix $\Gamma\mu^{-1}\Gamma$ ($\Gamma\kappa^{-1}\Gamma$) with respect to κ (μ). These eigenbivectors $\boldsymbol{E}$ ($\boldsymbol{H}$) may all be written as linear combinations of any two non parallel eigenbivectors $\boldsymbol{E}_1$, $\boldsymbol{E}_2$ ($\boldsymbol{H}_1$, $\boldsymbol{H}_2$). There are thus two wave solutions $\boldsymbol{D}_1, \boldsymbol{E}_1, \boldsymbol{B}_1, \boldsymbol{H}_1$ and $\boldsymbol{D}_2, \boldsymbol{E}_2, \boldsymbol{B}_2, \boldsymbol{H}_2$ corresponding to the same slowness $\boldsymbol{S} = N\boldsymbol{C}$, and any linear combination $\boldsymbol{D} = a\boldsymbol{D}_1 + b\boldsymbol{D}_2$, $\boldsymbol{E} = a\boldsymbol{E}_1 + b\boldsymbol{E}_2$, etc., with possibly complex coefficients a and b, is also a wave solution with the same slowness.

The eigenbivectors $\boldsymbol{E}_1$, $\boldsymbol{E}_2$ ($\boldsymbol{H}_1$, $\boldsymbol{H}_2$) may always be chosen to be orthogonal with respect to $\kappa(\mu)$ so that the orthogonality relations (4.15)–(4.18) hold.

Writing (3.1 *c*) for $\boldsymbol{E}_1$, $\boldsymbol{B}_1$ and taking the dot product with $\boldsymbol{H}_2$, writing (3.1 *d*) for $\boldsymbol{H}_2$, $\boldsymbol{D}_2$ and taking the dot product with $\boldsymbol{E}_2$, and adding, we obtain

$$\boldsymbol{D}_2 \cdot \boldsymbol{E}_1 = \boldsymbol{B}_1 \cdot \boldsymbol{H}_2, \tag{4.23}$$

as both wave solutions have the same slowness $\boldsymbol{S}$. Thus the orthogonality relations (4.15) and (4.17) imply the orthogonality relations (4.16) and (4.18) and vice versa. So, the two wave solutions may always be chosen to be such that they satisfy all the orthogonality relations of §4.2. Then, as a consequence, they also satisfy (4.19) and (4.20), so that the bivectors $\boldsymbol{D}_1$ and $\boldsymbol{B}_2$ are parallel, as are the bivectors $\boldsymbol{D}_2$ and $\boldsymbol{B}_1$.

Moreover, among all the linear combinations of two non parallel bivectors, there is always a linear bivector, that is a real vector up to a possibly complex scalar factor (see Appendix C). Thus $\boldsymbol{D}_1$, $\boldsymbol{E}_1$ ($\boldsymbol{B}_1$, $\boldsymbol{H}_1$) may always be chosen to be linear bivectors, and then $\boldsymbol{B}_2$, $\boldsymbol{H}_2$ ($\boldsymbol{D}_2$, $\boldsymbol{E}_2$) are also linear bivectors when the second wave solution is chosen to be such that all the orthogonality relations of §4.2 hold.

So, corresponding to the double root N^{-2} there are two wave solutions satisfying the orthogonality relations of §4.2, one with the fields $\mathbf{D}_1$ and $\mathbf{E}_1$ linearly polarized ($\mathbf{B}_1$ and $\mathbf{H}_1$

being then in general elliptically polarized) and the other with the fields $\mathbf{B}_2$ and $\mathbf{H}_2$ linearly polarized ($\mathbf{D}_2$ and $\mathbf{E}_2$ being then in general elliptically polarized). Any linear combination with coefficients a and b of the amplitudes of these two waves defines a wave with the same slowness $\boldsymbol{S} = N\boldsymbol{C}$. There are two values of the ratio b/a such that $\boldsymbol{E}$ and $\boldsymbol{H}$ are isotropic with respect to κ and μ respectively, and thus such that (4.21) and (4.22) are valid. These values are given by

$$\frac{b}{a} = \pm\mathrm{i}\frac{\boldsymbol{E}_1^{\mathrm{T}}\kappa\boldsymbol{E}_1}{\boldsymbol{E}_2^{\mathrm{T}}\kappa\boldsymbol{E}_2} = \pm\mathrm{i}\frac{\boldsymbol{H}_1^{\mathrm{T}}\mu\boldsymbol{H}_1}{\boldsymbol{H}_2^{\mathrm{T}}\mu\boldsymbol{H}_2} = \pm\mathrm{i}\frac{\sigma_1}{\sigma_2}, \tag{4.24}$$

where σ_1 and σ_2 are the values of σ defined by (4.4) for the two wave solutions.

5. Energy density and energy flux

At a point $\boldsymbol{x}$ in the crystal the energy flux (Poynting) vector $\mathbf{R}$ and the energy density $\mathbf{W}$ are defined by

$$\mathbf{R} = \mathbf{E}\times\mathbf{H}, \quad \mathbf{W} = \tfrac{1}{2}(\mathbf{D}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{H}). \tag{5.1}$$

They fluctuate in time so that to get a measure of energy propagation and energy density we calculate their mean values $\tilde{\mathbf{R}}, \tilde{\mathbf{W}}$ over a cycle:

$$\tilde{\mathbf{R}} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\mathbf{R}\,\mathrm{d}t, \quad \tilde{\mathbf{W}} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\mathbf{W}\,\mathrm{d}t. \tag{5.2}$$

In this section we first derive general results for $\tilde{\mathbf{R}}$ and $\tilde{\mathbf{W}}$ for inhomogeneous wave propagation. Next we consider the case when several waves may propagate with the same slowness (see §4.3, case 2).

In dealing with a wave of the form (2.7) it follows that (Hayes 1980)

$$\tilde{\mathbf{R}} = \boldsymbol{R}\exp(-2\omega\boldsymbol{S}^-\cdot\boldsymbol{x}), \quad \tilde{\mathbf{W}} = W\exp(-2\omega\boldsymbol{S}^-\cdot\boldsymbol{x}), \tag{5.3}$$

where $\boldsymbol{R}$ and W are given by

$$\boldsymbol{R} = \tfrac{1}{2}(\boldsymbol{E}\times\bar{\boldsymbol{H}})^+ = \tfrac{1}{4}(\boldsymbol{E}\times\bar{\boldsymbol{H}}+\bar{\boldsymbol{E}}\times\boldsymbol{H}), \tag{5.4}$$

and

$$W = \tfrac{1}{4}(\boldsymbol{D}\cdot\bar{\boldsymbol{E}}+\boldsymbol{B}\cdot\bar{\boldsymbol{H}}) = \tfrac{1}{4}(\boldsymbol{E}^{\mathrm{T}}\kappa\bar{\boldsymbol{E}}+\boldsymbol{H}^{\mathrm{T}}\mu\bar{\boldsymbol{H}}) = \tfrac{1}{4}(\boldsymbol{D}^{\mathrm{T}}\kappa^{-1}\bar{\boldsymbol{D}}+\boldsymbol{B}^{\mathrm{T}}\mu^{-1}\bar{\boldsymbol{B}}). \tag{5.5}$$

We call $\boldsymbol{R}$ and W the 'weighted energy flux' and 'weighted energy density' respectively. Because κ and μ (κ^{-1} and μ^{-1}) are real symmetric positive definite, the weighted energy density W is positive.

5.1. *General results*

In has been shown (Hayes 1980) for a train of inhomogeneous electromagnetic plane waves propagating with slowness bivector $\boldsymbol{S}$ in a crystal characterized by the constitutive equations (2.5) that

$$\boldsymbol{R}\cdot\boldsymbol{S}^+ = W, \quad \boldsymbol{R}\cdot\boldsymbol{S}^- = 0. \tag{5.6}$$

A geometrical interpretation of (5.6) is given in Hayes (1987). For each wave solution that we obtain in the sequel we will compute $\boldsymbol{R}$ and W and then use equations (5.6) as a check on their validity.

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Moreover, using the identity (A 3) of Appendix A, we obtain

$$\begin{aligned}\boldsymbol{R}^{\mathrm{T}}\mu^{-1}(\boldsymbol{S}^{+}\times\boldsymbol{S}^{-}) &= \tfrac{1}{4}(\boldsymbol{E}\times\bar{\boldsymbol{H}}+\bar{\boldsymbol{E}}\times\boldsymbol{H})^{\mathrm{T}}\mu^{-1}(\boldsymbol{S}\times\bar{\boldsymbol{S}})\\ &= \tfrac{1}{4}\mathrm{i}\,(\det\mu)^{-1}[(\boldsymbol{S}^{\mathrm{T}}\mu\boldsymbol{E})\,(\bar{\boldsymbol{S}}^{\mathrm{T}}\mu\bar{\boldsymbol{H}})-(\boldsymbol{S}^{\mathrm{T}}\mu\bar{\boldsymbol{H}})\,(\bar{\boldsymbol{S}}^{\mathrm{T}}\mu\boldsymbol{E})\\ &\quad+(\boldsymbol{S}^{\mathrm{T}}\mu\bar{\boldsymbol{E}})\,(\bar{\boldsymbol{S}}^{\mathrm{T}}\mu\boldsymbol{H})-(\boldsymbol{S}^{\mathrm{T}}\mu\boldsymbol{H})\,(\bar{\boldsymbol{S}}^{\mathrm{T}}\mu\bar{\boldsymbol{E}})].\end{aligned}\tag{5.7}$$

From (3.1 b) and (3.2), it follows that $\boldsymbol{S}^{\mathrm{T}}\mu\boldsymbol{H}=\boldsymbol{S}\cdot\boldsymbol{B}=0$, and thus (5.7) reduces to

$$\boldsymbol{R}^{\mathrm{T}}\mu^{-1}(\boldsymbol{S}^{+}\times\boldsymbol{S}^{-}) = \tfrac{1}{2}\det\mu^{-1}[(\boldsymbol{S}^{\mathrm{T}}\mu\bar{\boldsymbol{H}})\,(\bar{\boldsymbol{S}}^{\mathrm{T}}\mu\boldsymbol{H})]^{-}.\tag{5.8}$$

Similarly, by interchanging the rôles of κ and μ, we also obtain

$$\boldsymbol{R}^{\mathrm{T}}\kappa^{-1}(\boldsymbol{S}^{+}\times\boldsymbol{S}^{-}) = \tfrac{1}{2}\det\kappa^{-1}[(\boldsymbol{S}^{\mathrm{T}}\kappa\bar{\boldsymbol{H}})\,(\bar{\boldsymbol{S}}^{\mathrm{T}}\kappa\boldsymbol{E})]^{-},\tag{5.9}$$

and hence

$$\boldsymbol{R}^{\mathrm{T}}\{(\det\mu)\,\mu^{-1}+(\det\kappa)\,\kappa^{-1}\}\,\boldsymbol{S}^{+}\times\boldsymbol{S}^{-} = \tfrac{1}{2}[(\boldsymbol{S}^{\mathrm{T}}\mu\bar{\boldsymbol{H}})\,(\bar{\boldsymbol{S}}^{\mathrm{T}}\mu\boldsymbol{E})+(\boldsymbol{S}^{\mathrm{T}}\kappa\bar{\boldsymbol{H}})\,(\bar{\boldsymbol{S}}^{\mathrm{T}}\kappa\boldsymbol{E})]^{-}.\tag{5.10}$$

We observe that when the crystal is magnetically isotropic, i.e. when $\mu_{ij}=\mu\delta_{ij}$, (5.8) reduces to

$$\boldsymbol{R}\cdot(\boldsymbol{S}^{+}\times\boldsymbol{S}^{-}) = \tfrac{1}{2}[(\boldsymbol{S}\cdot\bar{\boldsymbol{H}})\,(\bar{\boldsymbol{S}}\cdot\boldsymbol{E})]^{-},\tag{5.11}$$

which was obtained in Hayes (1987). When the crystal is electrically isotropic, i.e. when $\kappa_{ij}=\kappa\delta_{ij}$, we recover equation (5.11) from equation (5.9). Thus, the result (5.11) holds whether the crystal is magnetically isotropic (but electrically anisotropic) or electrically isotropic (but magnetically anisotropic).

We also note using (3.1), (3.2) and the identity (A 2) of Appendix A that

$$\begin{aligned}\boldsymbol{E}\times\bar{\boldsymbol{H}} = \boldsymbol{E}\times\mu^{-1}(\bar{\boldsymbol{S}}\times\bar{\boldsymbol{E}}) &= (\det\mu)^{-1}\boldsymbol{E}\times(\mu\bar{\boldsymbol{S}}\times\mu\bar{\boldsymbol{E}})\\ &= \det\mu^{-1}\{(\boldsymbol{E}^{\mathrm{T}}\mu\bar{\boldsymbol{E}})\,\mu\bar{\boldsymbol{S}}-(\bar{\boldsymbol{S}}^{\mathrm{T}}\mu\boldsymbol{E})\,\mu\bar{\boldsymbol{E}}\},\end{aligned}\tag{5.12}$$

and hence

$$2\boldsymbol{R} = \det\mu^{-1}\{(\boldsymbol{E}^{\mathrm{T}}\mu\bar{\boldsymbol{E}})\,\mu\boldsymbol{S}^{+}-[(\boldsymbol{S}^{\mathrm{T}}\mu\bar{\boldsymbol{E}})\,\mu\boldsymbol{E}]^{+}\}.\tag{5.13}$$

Similarly, we have

$$2\boldsymbol{R} = \det\kappa^{-1}\{(\boldsymbol{H}^{\mathrm{T}}\kappa\bar{\boldsymbol{H}})\,\kappa\boldsymbol{S}^{+}-[(\boldsymbol{S}^{\mathrm{T}}\kappa\bar{\boldsymbol{H}})\,\kappa\boldsymbol{H}]^{+}\}.\tag{5.14}$$

Further using (3.1 c) and the identity (A 3) of Appendix A, we obtain

$$\begin{aligned}4W &= \boldsymbol{E}^{\mathrm{T}}\kappa\bar{\boldsymbol{E}}+(\boldsymbol{S}\times\boldsymbol{E})^{\mathrm{T}}\mu^{-1}(\bar{\boldsymbol{S}}\times\bar{\boldsymbol{E}})\\ &= \boldsymbol{E}^{\mathrm{T}}\kappa\bar{\boldsymbol{E}}+\det\mu^{-1}[(\boldsymbol{S}^{\mathrm{T}}\mu\bar{\boldsymbol{S}})\,(\boldsymbol{E}^{\mathrm{T}}\mu\bar{\boldsymbol{E}})-(\boldsymbol{S}^{\mathrm{T}}\mu\bar{\boldsymbol{E}})\,(\bar{\boldsymbol{S}}^{\mathrm{T}}\mu\boldsymbol{E})],\end{aligned}\tag{5.15}$$

and similarly

$$\begin{aligned}4W &= \boldsymbol{H}^{\mathrm{T}}\mu\bar{\boldsymbol{H}}+(\boldsymbol{S}\times\boldsymbol{H})^{\mathrm{T}}\kappa^{-1}(\bar{\boldsymbol{S}}\times\bar{\boldsymbol{H}})\\ &= \boldsymbol{H}^{\mathrm{T}}\mu\bar{\boldsymbol{H}}+\det\kappa^{-1}[(\boldsymbol{S}^{\mathrm{T}}\kappa\bar{\boldsymbol{S}})\,(\boldsymbol{H}^{\mathrm{T}}\kappa\bar{\boldsymbol{H}})-(\boldsymbol{S}^{\mathrm{T}}\kappa\bar{\boldsymbol{H}})\,(\bar{\boldsymbol{S}}^{\mathrm{T}}\kappa\boldsymbol{H})].\end{aligned}\tag{5.16}$$

But, using the propagation condition (3.10) and (3.11), we note that

$$\boldsymbol{E}^{\mathrm{T}}\kappa\bar{\boldsymbol{E}} = \det\mu^{-1}\{(\boldsymbol{S}^{\mathrm{T}}\mu\boldsymbol{S})\,(\boldsymbol{E}^{\mathrm{T}}\mu\bar{\boldsymbol{E}})-(\boldsymbol{S}^{\mathrm{T}}\mu\boldsymbol{E})\,(\boldsymbol{S}^{\mathrm{T}}\mu\bar{\boldsymbol{E}})\},\tag{5.17}$$

$$\boldsymbol{H}^{\mathrm{T}}\mu\bar{\boldsymbol{H}} = \det\kappa^{-1}\{(\boldsymbol{S}^{\mathrm{T}}\kappa\boldsymbol{S})\,(\boldsymbol{H}^{\mathrm{T}}\kappa\bar{\boldsymbol{H}})-(\boldsymbol{S}^{\mathrm{T}}\kappa\boldsymbol{H})\,(\boldsymbol{S}^{\mathrm{T}}\kappa\bar{\boldsymbol{H}})\},\tag{5.18}$$

and hence (5.15) and (5.16) become

$$2W = \det\mu^{-1}\{(\boldsymbol{S}^{\mathrm{T}}\mu\boldsymbol{S}^{+})\,(\boldsymbol{E}^{\mathrm{T}}\mu\bar{\boldsymbol{E}})-(\boldsymbol{E}^{\mathrm{T}}\mu\boldsymbol{S}^{+})\,(\boldsymbol{S}^{\mathrm{T}}\mu\bar{\boldsymbol{E}})\},\tag{5.19}$$

and

$$2W = \det\kappa^{-1}\{(\boldsymbol{S}^{\mathrm{T}}\kappa\boldsymbol{S}^{+})\,(\boldsymbol{H}^{\mathrm{T}}\kappa\bar{\boldsymbol{H}}) - (\boldsymbol{H}^{\mathrm{T}}\kappa\boldsymbol{S}^{+})\,(\boldsymbol{S}^{\mathrm{T}}\kappa\bar{\boldsymbol{H}})\}. \tag{5.20}$$

Taking the dot products of the right-hand sides of (5.13) and (5.14) with the slowness bivector $\boldsymbol{S}$, we obtain the right-hand sides of (5.19) and (5.20) respectively. Thus $\boldsymbol{R}\cdot\boldsymbol{S} = W$, and we recover (5.6) because $\boldsymbol{R}$ and W are real.

5.2. *Energy density and energy flux for waves with a common slowness*

It is of interest to consider here the case when two different waves may propagate with the same slowness (see §4.3, case 2). Let $\boldsymbol{D}_1$, $\boldsymbol{E}_1$, $\boldsymbol{B}_1$, $\boldsymbol{H}_1$ (with $\boldsymbol{D}_1$ and $\boldsymbol{E}_1$ linear bivectors) and $\boldsymbol{D}_2$, $\boldsymbol{E}_2$, $\boldsymbol{B}_2$, $\boldsymbol{H}_2$ (with $\boldsymbol{B}_2$ and $\boldsymbol{H}_2$ linear bivectors) be the amplitudes of the two waves propagating with the same slowness $\boldsymbol{S} = \boldsymbol{S}^+ + \mathrm{i}\boldsymbol{S}^-$ and satisfying the orthogonality relations of §4.2. Let the weighted energy densities and energy flux vectors for the two waves be denoted by W_1, $\boldsymbol{R}_1$ and W_2, $\boldsymbol{R}_2$. Then (5.6) holds for each wave:

$$\left.\begin{aligned} \boldsymbol{R}_1\cdot\boldsymbol{S}^+ = W_1, \quad \boldsymbol{R}_2\cdot\boldsymbol{S}^+ = W_2,\\ \boldsymbol{R}_1\cdot\boldsymbol{S}^- = 0, \quad \boldsymbol{R}_2\cdot\boldsymbol{S}^- = 0. \end{aligned}\right\} \tag{5.21}$$

Now, any linear combination $\boldsymbol{D} = a\boldsymbol{D}_1 + b\boldsymbol{D}_2$, $\boldsymbol{E} = a\boldsymbol{E}_1 + b\boldsymbol{E}_2$, etc., is also a wave with the same slowness. Let the weighted energy density and weighted energy flux for these combined waves be denoted by W and $\boldsymbol{R}$.

As $\boldsymbol{D}_1$, $\boldsymbol{E}_1$, $\boldsymbol{B}_2$, $\boldsymbol{H}_2$ may be taken to be real, we note, using (4.15)–(4.18), that

$$W = |a|^2 W_1 + |b|^2 W_2, \tag{5.22}$$

that is, there is no interaction term in the energy density. There may be an interaction term in the energy flux:

$$\boldsymbol{R} = |a|^2\boldsymbol{R}_1 + |b|^2\boldsymbol{R}_2 + \boldsymbol{I}, \tag{5.23}$$

where $\boldsymbol{I}$ is the interaction term. Then from (5.6), (5.22), (5.23),

$$|a|^2\boldsymbol{R}_1\cdot\boldsymbol{S}^+ + |b|^2\boldsymbol{R}_2\cdot\boldsymbol{S}^+ + \boldsymbol{I}\cdot\boldsymbol{S}^+ = |a|^2\boldsymbol{W}_1 + |b|^2\boldsymbol{W}_2, \tag{5.24}$$

$$|a|^2\boldsymbol{R}_1\cdot\boldsymbol{S}^- + |b|^2\boldsymbol{R}_2\cdot\boldsymbol{S}^- + \boldsymbol{I}\cdot\boldsymbol{S}^- = 0, \tag{5.25}$$

and using (5.21), we obtain

$$\boldsymbol{I}\cdot\boldsymbol{S}^+ = \boldsymbol{I}\cdot\boldsymbol{S}^- = 0. \tag{5.26}$$

Hence for inhomogeneous waves, $\boldsymbol{R}$ is of the form

$$\boldsymbol{R} = |a|^2\boldsymbol{R}_1 + |b|^2\boldsymbol{R}_2 + \nu\boldsymbol{S}^+\times\boldsymbol{S}^-, \tag{5.27}$$

where ν is some real scalar.

6. Universal relations

Universal relations are valid for all crystals described by the constitutive equations (2.5), independently of the choice of the constitutive tensors κ, μ. Here some universal relations involving the phase speeds and attenuation factors are obtained.

Let us denote by $N_1^{-2}(\boldsymbol{C})$, $N_2^{-2}(\boldsymbol{C})$ the two roots of the secular equation (3.15) or (3.16) corresponding to the bivector $\boldsymbol{C}$. From (3.15) and (3.16) it is clear that the sum of these roots is a quadratic form $Q(\boldsymbol{C})$ in the components of the bivector $\boldsymbol{C}$:

$$N_1^{-2}(\boldsymbol{C}) + N_2^{-2}(\boldsymbol{C}) = \boldsymbol{C}^{\mathrm{T}}\Phi\boldsymbol{C} = Q(\boldsymbol{C}), \tag{6.1}$$

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where $\boldsymbol{\Phi}$ is the symmetric tensor given by

$$\boldsymbol{\Phi} = (\det\kappa)^{-1}[\mathrm{tr}(\kappa\mu^{-1})\,\kappa-\kappa\mu^{-1}\kappa], \tag{6.2}$$

or equivalently by

$$\boldsymbol{\Phi} = (\det\mu)^{-1}[\mathrm{tr}(\mu\kappa^{-1})\,\mu-\mu\kappa^{-1}\mu]. \tag{6.3}$$

In the case of homogeneous waves, the bivector $\boldsymbol{C}$ is replaced by a real unit vector $\hat{\boldsymbol{n}}$, and $c_1^2(\hat{\boldsymbol{n}}) = N_1^{-2}(\hat{\boldsymbol{n}})$, $c_2^2(\hat{\boldsymbol{n}}) = N_2^{-2}(\hat{\boldsymbol{n}})$ are the phase speeds of the waves that may propagate along $\hat{\boldsymbol{n}}$. Then (6.1) becomes

$$c_1^2(\hat{\boldsymbol{n}})+c_2^2(\hat{\boldsymbol{n}}) = \hat{\boldsymbol{n}}^{\mathrm{T}}\boldsymbol{\Phi}\hat{\boldsymbol{n}} = Q(\hat{\boldsymbol{n}}). \tag{6.4}$$

Now, using (2.12*b*), the quadratic form $Q(\boldsymbol{C}) = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{C}$ may be written as

$$\begin{aligned} Q(m\hat{\boldsymbol{m}}+\mathrm{i}\hat{\boldsymbol{n}}) &= m^2\hat{\boldsymbol{m}}^{\mathrm{T}}\boldsymbol{\Phi}\hat{\boldsymbol{m}}-\hat{\boldsymbol{n}}^{\mathrm{T}}\boldsymbol{\Phi}\hat{\boldsymbol{n}}+2\mathrm{i}m\hat{\boldsymbol{m}}^{\mathrm{T}}\boldsymbol{\Phi}\hat{\boldsymbol{n}} \\ &= m^2Q(\hat{\boldsymbol{m}})-Q(\hat{\boldsymbol{n}})+\mathrm{i}m\,[Q\surd\tfrac{1}{2}(\hat{\boldsymbol{m}}+\hat{\boldsymbol{n}})-Q\surd\tfrac{1}{2}(\hat{\boldsymbol{m}}-\hat{\boldsymbol{n}})]. \end{aligned} \tag{6.5}$$

Together with (6.1) and (6.4), this identity gives the universal relation

$$\sum_{\alpha=1}^{2} N_\alpha^{-2}(m\hat{\boldsymbol{m}}+\mathrm{i}\hat{\boldsymbol{n}}) = m^2\sum_{\alpha=1}^{2} c_\alpha^2(\hat{\boldsymbol{m}})-\sum_{\alpha=1}^{2} c_\alpha^2(\hat{\boldsymbol{n}})+\mathrm{i}m\sum_{\alpha=1}^{2}[c_\alpha^2\surd\tfrac{1}{2}(\hat{\boldsymbol{m}}+\hat{\boldsymbol{n}})-c_\alpha^2\surd\tfrac{1}{2}(\hat{\boldsymbol{m}}-\hat{\boldsymbol{n}})]. \tag{6.6}$$

Taking the real and imaginary parts of (6.5) we obtain, in the same way, the universal relations

$$\sum_{\alpha=1}^{2} N_\alpha^{-2}(m\hat{\boldsymbol{m}}+\mathrm{i}\hat{\boldsymbol{n}})+\sum_{\alpha=1}^{2} N_\alpha^{-2}(m\hat{\boldsymbol{m}}-\mathrm{i}\hat{\boldsymbol{n}}) = 2m^2\sum_{\alpha=1}^{2} c_\alpha^2(\hat{\boldsymbol{m}})-2\sum_{\alpha=1}^{2} c_\alpha^2(\hat{\boldsymbol{n}}), \tag{6.7}$$

and

$$\sum_{\alpha=1}^{2} N_\alpha^{-2}(m\hat{\boldsymbol{m}}+\mathrm{i}\hat{\boldsymbol{n}})-\sum_{\alpha=1}^{2} N_\alpha^{-2}(m\hat{\boldsymbol{m}}-\mathrm{i}\hat{\boldsymbol{n}}) = 2\mathrm{i}m\sum_{\alpha=1}^{2}\left[c_\alpha^2\left(\frac{\hat{\boldsymbol{m}}+\hat{\boldsymbol{n}}}{\surd 2}\right)-c_\alpha^2\left(\frac{\hat{\boldsymbol{m}}-\hat{\boldsymbol{n}}}{\surd 2}\right)\right]. \tag{6.8}$$

The universal relations (6.6)–(6.8) are the same as those derived in Hayes (1987) for the case of magnetically isotropic but electrically anisotropic crystals.

Now, let $\hat{\boldsymbol{C}}_1$, $\hat{\boldsymbol{C}}_2$, $\hat{\boldsymbol{C}}_3$ be any orthonormal triad of bivectors or vectors:

$$\hat{\boldsymbol{C}}_\beta\cdot\hat{\boldsymbol{C}}_\gamma = \delta_{\beta\gamma} \quad (\beta,\gamma = 1,2,3). \tag{6.9}$$

From (6.1) it follows that

$$\sum_{\alpha=1}^{2}[N_\alpha^{-2}(\hat{\boldsymbol{C}}_1)+N_\alpha^{-2}(\hat{\boldsymbol{C}}_2)+N_\alpha^{-2}(\hat{\boldsymbol{C}}_3)] = \Phi_{ij}(\hat{C}_{1i}\hat{C}_{1j}+\hat{C}_{2i}\hat{C}_{2j}+\hat{C}_{3i}\hat{C}_{3j}). \tag{6.10}$$

Because

$$\hat{C}_{1i}\hat{C}_{1j}+\hat{C}_{2i}\hat{C}_{2j}+\hat{C}_{3i}\hat{C}_{3j} = \delta_{ij}, \tag{6.11}$$

we obtain

$$\begin{aligned} \sum_{\beta=1}^{3}\left(\sum_{\alpha=1}^{2} N_\alpha^{-2}(\hat{\boldsymbol{C}}_\beta)\right) &= \mathrm{tr}\,\boldsymbol{\Phi} = (\det\kappa)^{-1}[(\mathrm{tr}\,\kappa)\,\mathrm{tr}(\kappa\mu^{-1})-\mathrm{tr}(\kappa^2\mu^{-1})] \\ &= (\det\mu)^{-1}[(\mathrm{tr}\,\mu)\,\mathrm{tr}(\mu\kappa^{-1})-\mathrm{tr}(\mu^2\kappa^{-1})], \end{aligned} \tag{6.12}$$

and using the Cayley–Hamilton theorem in order to eliminate κ^2 or μ^2,

$$\sum_{\beta=1}^{3}\left(\sum_{\alpha=1}^{2} N_\alpha^{-2}(\hat{\boldsymbol{C}}_\beta)\right) = \mathrm{tr}(\kappa^{-1})\,\mathrm{tr}(\mu^{-1})-\mathrm{tr}(\kappa^{-1}\mu^{-1}). \tag{6.13}$$

Thus, the sum

$$\sum_{\beta=1}^{3}\sum_{\alpha=1}^{2} N_\alpha^{-2}(\hat{\boldsymbol{C}}_\beta)$$

is the same for every orthonormal triad of bivectors or vectors. In particular (6.13) holds when $\hat{\boldsymbol{C}}_\beta = \hat{\boldsymbol{p}}_\beta$, where $\hat{\boldsymbol{p}}_\beta$ ($\beta = 1, 2, 3$), denotes any orthonormal triad of real vectors:

$$\sum_{\beta=1}^{3}\left(\sum_{\alpha=1}^{2} c_\alpha^2(\hat{\boldsymbol{p}}_\beta)\right) = \mathrm{tr}(\kappa^{-1})\,\mathrm{tr}(\mu^{-1})-\mathrm{tr}(\kappa^{-1}\mu^{-1}). \tag{6.14}$$

As in Hayes (1987) we then derive, using (6.7),

$$\sum_{\alpha=1}^{2}\sum_{\beta,\gamma=1}^{3}[N_\alpha^{-2}(r\hat{\boldsymbol{p}}_\beta+\mathrm{i}\hat{\boldsymbol{p}}_\gamma)+N_\alpha^{-2}(r\hat{\boldsymbol{p}}_\beta-\mathrm{i}\hat{\boldsymbol{p}}_\gamma)]$$

$$=2(r^2-1)\sum_{\beta=1}^{3}\sum_{\alpha=1}^{2}c_\alpha^2(\hat{\boldsymbol{p}}_\beta)=2(r^2-1)\,[\mathrm{tr}\,(\kappa^{-1})\,\mathrm{tr}\,(\mu^{-1})-\mathrm{tr}\,(\kappa^{-1}\mu^{-1})], \tag{6.15}$$

where r is any real number.

Finally, suppose $\hat{\boldsymbol{C}}_1$ and $\hat{\boldsymbol{C}}_2$ are a pair of orthogonal unit bivectors, and let $\hat{\boldsymbol{C}}_1'$ and $\hat{\boldsymbol{C}}_2'$ be any pair of orthogonal unit bivectors 'coplanar' with $\hat{\boldsymbol{C}}_1$ and $\hat{\boldsymbol{C}}_2$:

$$\hat{\boldsymbol{C}}_1'=\cos\theta\hat{\boldsymbol{C}}_1+\sin\theta\hat{\boldsymbol{C}}_2,\quad \hat{\boldsymbol{C}}_2'=-\sin\theta\hat{\boldsymbol{C}}_1+\cos\theta\hat{\boldsymbol{C}}_2. \tag{6.16}$$

Then using (6.1) we obtain as in Hayes (1987) the universal relation

$$\sum_{\alpha=1}^{2}[N_\alpha^{-2}(\hat{\boldsymbol{C}}_1)+N_\alpha^{-2}(\hat{\boldsymbol{C}}_2)]=\sum_{\alpha=1}^{2}[N_\alpha^{-2}(\hat{\boldsymbol{C}}_1')+N_\alpha^{-2}(\hat{\boldsymbol{C}}_2')]. \tag{6.17}$$

7. Homogeneous waves

In this section we restrict attention to homogeneous plane wave solutions, that is solutions for which the planes of constant phase are also planes of constant amplitude. The emphasis is on the geometrical point of view. It is shown here for non repeated eigenvalues that the waves are linearly polarized. For such waves propagating in the direction $\hat{\boldsymbol{n}}$ it is shown that the amplitude vectors $\boldsymbol{B}$ and $\boldsymbol{D}$ must lie along the pair of common conjugate directions of the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane $\Pi(\hat{\boldsymbol{n}})$ whose normal is $\hat{\boldsymbol{n}}$. (In general any pair of coplanar ellipses possesses one pair of common conjugate directions. Of course if the ellipses are similar and similarly situated they possess an infinite number of pairs of common conjugate directions.) The case of repeated eigenvalues is deferred to §8.

To deal with homogeneous waves we assume that the slowness bivector $\boldsymbol{S}$ is given by (2.13), so that the skew symmetric tensor Γ defined by (2.14) with $\boldsymbol{C}=\hat{\boldsymbol{n}}$ is now real.

As $\Gamma\mu^{-1}\Gamma$ and $\Gamma\kappa^{-1}\Gamma$ are now real symmetric matrices, their eigenvalues $(-N^{-2})$ with respect to κ and μ respectively, are real. Also, for non-repeated eigenvalues, the eigenbivectors $\boldsymbol{E}$ and $\boldsymbol{H}$, solutions respectively of the eigenvalue problems (3.4) and (3.6) are linear bivectors, that is (possibly complex) scalar multiples of real vectors. Of course, from the constitutive equations (3.2) it then follows trivially that $\boldsymbol{D}$ and $\boldsymbol{B}$ are also linear bivectors. In the case of repeated eigenvalues, however, $\boldsymbol{B}$, $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{H}$ may be bivectors.

Also, taking the dot product of (3.4) with $\bar{\boldsymbol{E}}$ and the dot product of (3.6), with $\bar{\boldsymbol{H}}$, we obtain, because Γ is skew symmetric and real,

$$(\overline{\Gamma\boldsymbol{E}})^{\mathrm{T}}\mu^{-1}(\Gamma\boldsymbol{E})=N^{-2}\bar{\boldsymbol{E}}^{\mathrm{T}}\kappa\boldsymbol{E},\quad (\overline{\Gamma\boldsymbol{H}})^{\mathrm{T}}\kappa^{-1}(\Gamma\boldsymbol{H})=N^{-2}\bar{\boldsymbol{H}}^{\mathrm{T}}\mu\boldsymbol{H}. \tag{7.1}$$

As κ, μ are positive definite, N^{-2} is positive (if not zero), and hence N is real, which means that the homogeneous waves are not attenuated.

Turning now to the orthogonality relations of §4.1, the equation (4.1) now means trivially that the direction of $\boldsymbol{E}$ is orthogonal to the direction of $\boldsymbol{B}$ and the direction of $\boldsymbol{D}$ is orthogonal to the direction of $\boldsymbol{H}$. Also, from (3.1 a, b) $\boldsymbol{D}$ and $\boldsymbol{B}$ are in the plane $\Pi(\hat{\boldsymbol{n}})$ whose normal is $\hat{\boldsymbol{n}}$. Equation (4.2) means that the linear bivectors $\boldsymbol{D}$ and $\boldsymbol{B}$ must be along common conjugate directions of the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane $\Pi(\hat{\boldsymbol{n}})$. There is only one pair of such directions if these elliptical sections are not similar and similarly situated, and in §8, it will be shown that these elliptical sections are similar and similarly situated if and only if the secular equation (3.15) or (3.16) has a double root. Thus, we have here two possibilities for the directions of $\boldsymbol{D}$ and $\boldsymbol{B}$ corresponding to the two different roots of the secular equation (see figure 1*a*).

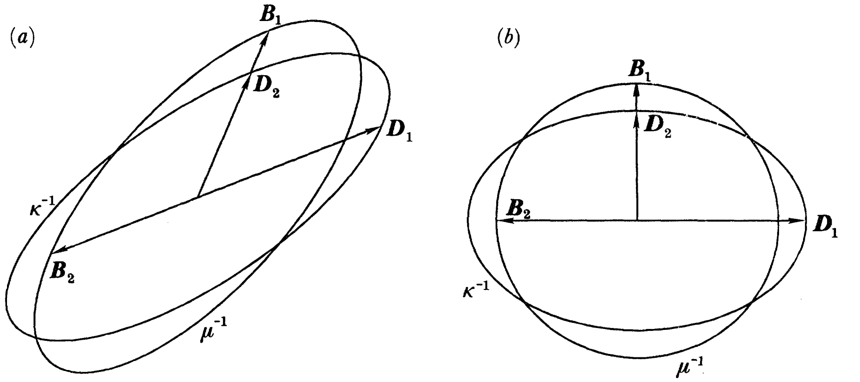

FIGURE 1. The disposition of $\boldsymbol{D}_1$ ($\boldsymbol{B}_2$) and $\boldsymbol{B}_1$ ($\boldsymbol{D}_2$) corresponding to the two homogeneous plane waves propagating along a direction $\hat{\boldsymbol{n}}$ which is not along a (generalized) optic axis. In (*a*), both μ and κ are anisotropic: $\boldsymbol{D}_1$ ($\boldsymbol{B}_2$) and $\boldsymbol{B}_1$ ($\boldsymbol{D}_2$) are along the common conjugate directions of the elliptical sections of the μ^{-1} and κ^{-1}-metric ellipsoids by the plane $\Pi(\hat{\boldsymbol{n}})$. In (*b*), μ is isotropic: $\boldsymbol{D}_1$ ($\boldsymbol{B}_2$) and $\boldsymbol{B}_1$ ($\boldsymbol{D}_2$) are along the principal axes of the elliptical section of the κ^{-1}-metric ellipsoid by the plane $\Pi(\hat{\boldsymbol{n}})$.

The direction of $\boldsymbol{E}$ may be determined when the direction of $\boldsymbol{D}$ is known. For $\boldsymbol{E} = \kappa^{-1}\boldsymbol{D}$, and hence $\boldsymbol{E}$ is parallel to the normal to the κ^{-1}-metric ellipsoid at the point M where the radius in the direction of $\boldsymbol{D}$ intersects it. Similarly, because $\boldsymbol{H} = \mu^{-1}\boldsymbol{B}$, $\boldsymbol{H}$ is parallel to the normal to the μ^{-1}-metric ellipsoid at the point N where the radius in the direction of $\boldsymbol{B}$ intersects it.

Thus the directions of $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{H}$ are determined. As for their magnitudes, it follows from equation (4.5) that if the extremity of $\boldsymbol{D}$ (taken to be real) is on the κ^{-1}-metric ellipsoid, that is $\boldsymbol{D}^{\mathrm{T}}\kappa^{-1}\boldsymbol{D} = 1$, then the extremities of $\boldsymbol{E}$, $\boldsymbol{H}$ and $\boldsymbol{B}$ (also real) are respectively on the κ, μ, μ^{-1}-metric ellipsoids (see figure 1*a*).

From the results of §4.2 for two different roots N_1^{-2} and N_2^{-2} of the secular equation the vectors $\boldsymbol{D}_1$ and $\boldsymbol{B}_2$ are parallel and so are the vectors $\boldsymbol{D}_2$ and $\boldsymbol{B}_1$, so confirming the geometrical interpretation obtained above. So, $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ ($\boldsymbol{B}_2$ and $\boldsymbol{B}_1$) must lie along the common conjugate directions of the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane $\Pi(\hat{\boldsymbol{n}})$. The contrast between this and the classical case (μ isotropic) is illustrated in figure 1.

Of course if $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ are conjugate radii of the κ^{-1}-metric ellipsoid, then $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ are conjugate radii of the κ-metric ellipsoid. This follows because if $\boldsymbol{D}_1^{\mathrm{T}}\kappa^{-1}\boldsymbol{D}_1 = 1$, $\boldsymbol{D}_2^{\mathrm{T}}\kappa^{-1}\boldsymbol{D}_2 = 1$, $\boldsymbol{D}_1^{\mathrm{T}}\kappa^{-1}\boldsymbol{D}_2 = 0$, then $\boldsymbol{E}_1^{\mathrm{T}}\kappa\boldsymbol{E}_1 = 1$, $\boldsymbol{E}_2^{\mathrm{T}}\kappa\boldsymbol{E}_2 = 1$, $\boldsymbol{E}_1^{\mathrm{T}}\kappa\boldsymbol{E}_2 = 0$. Then also $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are conjugate radii of the μ^{-1}-metric ellipsoid, and $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ are conjugate radii of the μ-metric ellipsoid.

Also, for homogeneous waves, it follows from (5.4) that the weighted energy flux vector $\boldsymbol{R}$ is orthogonal to the directions of $\boldsymbol{E}$ and $\boldsymbol{H}$. This means that the direction of $\boldsymbol{R}$ is parallel to the intersection of the tangent plane to the κ^{-1}-metric ellipsoid at the point M with the tangent plane to the μ^{-1}-metric ellipsoid at the point N.

From equation (4.13), it follows that the slowness of the wave propagating in the direction $\hat{\boldsymbol{n}}$ with amplitude $\boldsymbol{D}_1$ is given by

$$N_1^2\,(\det\mu)^{-1}(\hat{\boldsymbol{n}}^{\mathrm{T}}\mu\hat{\boldsymbol{n}})\,(\boldsymbol{D}_1^{\mathrm{T}}\kappa^{-1}\boldsymbol{D}_1) = \boldsymbol{D}_1^{\mathrm{T}}\mu^{-1}\boldsymbol{D}_1, \qquad (7.2)$$

and similarly for the wave with amplitude $\boldsymbol{D}_2$,

$$N_2^2\,(\det\mu)^{-1}(\hat{\boldsymbol{n}}^{\mathrm{T}}\mu\hat{\boldsymbol{n}})\,(\boldsymbol{D}_2^{\mathrm{T}}\kappa^{-1}\boldsymbol{D}_2) = \boldsymbol{D}_2^{\mathrm{T}}\mu^{-1}\boldsymbol{D}_2. \tag{7.3}$$

We denote by $\hat{\boldsymbol{d}}_1$ and $\hat{\boldsymbol{d}}_2$, the unit vectors along $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ respectively, that is the unit vectors along the common conjugate radii of the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane $\Pi(\hat{\boldsymbol{n}})$. We note that $\hat{\boldsymbol{n}}\sin\theta = \hat{\boldsymbol{d}}_1\times\hat{\boldsymbol{d}}_2$, where θ is the angle between $\hat{\boldsymbol{d}}_1$ and $\hat{\boldsymbol{d}}_2$.

Using the fact that $\boldsymbol{D}_1$ is parallel to $\boldsymbol{B}_2$, and that $\boldsymbol{D}_2$ is parallel to $\boldsymbol{B}_1$, we have from (7.2), (7.3) and (4.13*b*),

$$\left.\begin{aligned} N_1^{-2} &= (\det\mu)^{-1}(\hat{\boldsymbol{n}}^{\mathrm{T}}\mu\hat{\boldsymbol{n}})\,\frac{\hat{\boldsymbol{d}}_1^{\mathrm{T}}\kappa^{-1}\hat{\boldsymbol{d}}_1}{\hat{\boldsymbol{d}}_1^{\mathrm{T}}\mu^{-1}\hat{\boldsymbol{d}}_1} = (\det\kappa)^{-1}(\hat{\boldsymbol{n}}^{\mathrm{T}}\kappa\hat{\boldsymbol{n}})\,\frac{\hat{\boldsymbol{d}}_2^{\mathrm{T}}\mu^{-1}\hat{\boldsymbol{d}}_2}{\hat{\boldsymbol{d}}_2^{\mathrm{T}}\kappa^{-1}\hat{\boldsymbol{d}}_2},\\ N_2^{-2} &= (\det\mu)^{-1}(\hat{\boldsymbol{n}}^{\mathrm{T}}\mu\hat{\boldsymbol{n}})\,\frac{\hat{\boldsymbol{d}}_2^{\mathrm{T}}\kappa^{-1}\hat{\boldsymbol{d}}_2}{\hat{\boldsymbol{d}}_2^{\mathrm{T}}\mu^{-1}\hat{\boldsymbol{d}}_2} = (\det\kappa)^{-1}(\hat{\boldsymbol{n}}^{\mathrm{T}}\kappa\hat{\boldsymbol{n}})\,\frac{\hat{\boldsymbol{d}}_1^{\mathrm{T}}\mu^{-1}\hat{\boldsymbol{d}}_1}{\hat{\boldsymbol{d}}_1^{\mathrm{T}}\kappa^{-1}\hat{\boldsymbol{d}}_1}. \end{aligned}\right\} \tag{7.4}$$

To make the consistency of these relations transparent we note first that

$$\begin{aligned}\hat{\boldsymbol{n}}\cdot\{(\mu^{-1}\hat{\boldsymbol{d}}_1)\times(\mu^{-1}\hat{\boldsymbol{d}}_2)\}\sin\theta &= (\hat{\boldsymbol{d}}_1\times\hat{\boldsymbol{d}}_2)\cdot\{(\mu^{-1}\hat{\boldsymbol{d}}_1)\times(\mu^{-1}\hat{\boldsymbol{d}}_2)\}\\ &= (\hat{\boldsymbol{d}}_1^{\mathrm{T}}\mu^{-1}\hat{\boldsymbol{d}}_1)\,(\hat{\boldsymbol{d}}_2^{\mathrm{T}}\mu^{-1}\hat{\boldsymbol{d}}_2),\end{aligned} \tag{7.5}$$

because $\hat{\boldsymbol{d}}_1^{\mathrm{T}}\mu^{-1}\hat{\boldsymbol{d}}_2 = 0$. Also, from identity (A 1) of Appendix A,

$$\begin{aligned}\hat{\boldsymbol{n}}\cdot\{(\mu^{-1}\hat{\boldsymbol{d}}_1)\times(\mu^{-1}\hat{\boldsymbol{d}}_2)\} &= (\det\mu)^{-1}\{\hat{\boldsymbol{n}}\cdot\mu(\hat{\boldsymbol{d}}_1\times\hat{\boldsymbol{d}}_2)\}\\ &= (\det\mu)^{-1}(\hat{\boldsymbol{n}}^{\mathrm{T}}\mu\hat{\boldsymbol{n}})\sin\theta.\end{aligned} \tag{7.6}$$

Thus

$$\left.\begin{aligned}(\hat{\boldsymbol{d}}_1^{\mathrm{T}}\mu^{-1}\hat{\boldsymbol{d}}_1)\,(\hat{\boldsymbol{d}}_2^{\mathrm{T}}\mu^{-1}\hat{\boldsymbol{d}}_2) &= (\det\mu)^{-1}(\hat{\boldsymbol{n}}^{\mathrm{T}}\mu\hat{\boldsymbol{n}})\sin^2\theta,\\ (\hat{\boldsymbol{d}}_1^{\mathrm{T}}\kappa^{-1}\hat{\boldsymbol{d}}_1)\,(\hat{\boldsymbol{d}}_2^{\mathrm{T}}\kappa^{-1}\hat{\boldsymbol{d}}_2) &= (\det\kappa)^{-1}(\hat{\boldsymbol{n}}^{\mathrm{T}}\kappa\hat{\boldsymbol{n}})\sin^2\theta,\end{aligned}\right\} \tag{7.7}$$

the second of these being derived analogously to the first. The expressions (7.4) are clearly consistent.

Then, from (7.2), (7.3) and (7.7), we have

$$\left.\begin{aligned}N_1^2 &= (\hat{\boldsymbol{d}}_1^{\mathrm{T}}\kappa^{-1}\hat{\boldsymbol{d}}_1)^{-1}(\hat{\boldsymbol{d}}_2^{\mathrm{T}}\mu^{-1}\hat{\boldsymbol{d}}_2)^{-1}\sin^2\theta,\\ N_2^2 &= (\hat{\boldsymbol{d}}_2^{\mathrm{T}}\kappa^{-1}\hat{\boldsymbol{d}}_2)^{-1}(\hat{\boldsymbol{d}}_1^{\mathrm{T}}\mu^{-1}\hat{\boldsymbol{d}}_1)^{-1}\sin^2\theta.\end{aligned}\right\} \tag{7.8}$$

Now $(\hat{\boldsymbol{d}}_1^{\mathrm{T}}\kappa^{-1}\hat{\boldsymbol{d}}_1)^{-1}$ is the square of the length of the radius to the ellipsoid $\boldsymbol{x}^{\mathrm{T}}\kappa^{-1}\boldsymbol{x} = 1$ along $\hat{\boldsymbol{d}}_1$, and $(\hat{\boldsymbol{d}}_2^{\mathrm{T}}\mu^{-1}\hat{\boldsymbol{d}}_2)^{-1}$ is the square of the length of the radius to the ellipsoid $\boldsymbol{x}^{\mathrm{T}}\mu^{-1}\boldsymbol{x} = 1$ along $\hat{\boldsymbol{d}}_2$. Thus the slowness, N_1, of the wave propagating along $\hat{\boldsymbol{n}}$ with **D** amplitude along $\hat{\boldsymbol{d}}_1$, is equal to the area of the parallelogram formed by the radius along $\hat{\boldsymbol{d}}_1$ to the κ^{-1}-metric ellipsoid and the radius along $\hat{\boldsymbol{d}}_2$ to the μ^{-1}-metric ellipsoid. The slowness, N_2, of the wave propagating along $\hat{\boldsymbol{n}}$, with **D** amplitude along $\hat{\boldsymbol{d}}_2$, is equal to the area of the parallelogram formed by the radius along $\hat{\boldsymbol{d}}_2$ to the κ^{-1}-metric ellipsoid and the radius along $\hat{\boldsymbol{d}}_1$ to the μ^{-1}-metric ellipsoid.

Remark 1. *Magnetically isotropic crystals*

We retrieve well-known results (Born & Wolf 1980, p. 673) for the special case when the crystal is magnetically isotropic, that is $\mu = \mu 1$. Then the μ^{-1}-metric ellipsoid is a sphere of radius $\surd\mu$. The central sections of the μ^{-1} and κ^{-1}-metric ellipsoids by the plane $\Pi(\hat{\boldsymbol{n}})$ are respectively a circle of radius $\surd\mu$ and an ellipse, in general. These have common conjugate radii along the principal axes of the ellipse. Thus the amplitudes of the waves are along the principal axes of the ellipse. The slownesses of the waves are equal to $\sqrt{\mu}$ times the lengths of the principal semi-axes of the ellipse.

For the homogeneous wave propagating along $\hat{\boldsymbol{n}}$, with the amplitude $\boldsymbol{D}$ along $\hat{\boldsymbol{d}}_1$, we find, using $\hat{\boldsymbol{n}} = (\hat{\boldsymbol{d}}_1 \times \hat{\boldsymbol{d}}_2) \operatorname{cosec}\theta$,

$$\left.\begin{aligned} \boldsymbol{D}_1 &= \hat{\boldsymbol{d}}_1, \quad \boldsymbol{B}_1 = N_1(\hat{\boldsymbol{d}}_1^{\mathrm{T}} \kappa^{-1} \hat{\boldsymbol{d}}_1)\,(\operatorname{cosec}\theta)\,\hat{\boldsymbol{d}}_2, \\ \boldsymbol{E}_1 &= \kappa^{-1}\hat{\boldsymbol{d}}_1, \quad \boldsymbol{H}_1 = N_1(\hat{\boldsymbol{d}}_1^{\mathrm{T}} \kappa^{-1} \hat{\boldsymbol{d}}_1)\,(\operatorname{cosec}\theta)\,\mu^{-1}\hat{\boldsymbol{d}}_2, \\ 4W_1 &= \hat{\boldsymbol{d}}_1^{\mathrm{T}} \kappa^{-1} \hat{\boldsymbol{d}}_1 + (\hat{\boldsymbol{d}}_2^{\mathrm{T}} \mu^{-1} \hat{\boldsymbol{d}}_2)\,(\hat{\boldsymbol{d}}_1^{\mathrm{T}} \kappa^{-1} \hat{\boldsymbol{d}}_1)^2 N_1^2 \operatorname{cosec}^2\theta = 2\hat{\boldsymbol{d}}_1^{\mathrm{T}} \kappa^{-1} \hat{\boldsymbol{d}}_1, \\ 2\boldsymbol{R}_1 &= N_1 \operatorname{cosec}\theta(\hat{\boldsymbol{d}}_1^{\mathrm{T}} \kappa^{-1} \hat{\boldsymbol{d}}_1)\,\kappa^{-1}\boldsymbol{d}_1 \times \mu^{-1}\hat{\boldsymbol{d}}_2. \end{aligned}\right\} \tag{7.9}$$

Of course the amplitudes $\boldsymbol{D}_1, \boldsymbol{E}_1, \boldsymbol{B}_1, \boldsymbol{H}_1$ may be multiplied by an arbitrary scalar factor a. The corresponding weighted energy density W_1 and energy flux $\boldsymbol{R}_1$ are then multiplied by the factor $a\bar{a}$. This comment applies to any wave solution obtained in this paper and these factors will not be written explicitly. We check that

$$2N_1 \boldsymbol{R}_1 \cdot \hat{\boldsymbol{n}} = N_1^2 \operatorname{cosec}^2\theta(\hat{\boldsymbol{d}}_1^{\mathrm{T}} \kappa^{-1} \hat{\boldsymbol{d}}_1)^2 (\hat{\boldsymbol{d}}_2^{\mathrm{T}} \mu^{-1} \hat{\boldsymbol{d}}_2) = \hat{\boldsymbol{d}}_1^{\mathrm{T}} \kappa^{-1} \hat{\boldsymbol{d}}_1 = 2W_1,$$

so confirming (5.6).

Similarly, for the homogeneous wave propagating along $\hat{\boldsymbol{n}}$, with the amplitude $\boldsymbol{D}$ along $\hat{\boldsymbol{d}}_2$, we find,

$$\left.\begin{aligned} \boldsymbol{D}_2 &= \hat{\boldsymbol{d}}_2, \quad \boldsymbol{B}_2 = -N_2(\hat{\boldsymbol{d}}_2^{\mathrm{T}} \kappa^{-1} \hat{\boldsymbol{d}}_2)\,(\operatorname{cosec}\theta)\,\hat{\boldsymbol{d}}_1, \\ \boldsymbol{E}_2 &= \kappa^{-1}\hat{\boldsymbol{d}}_2, \quad \boldsymbol{H}_2 = -N_2(\hat{\boldsymbol{d}}_2^{\mathrm{T}} \kappa^{-1} \hat{\boldsymbol{d}}_2)\,(\operatorname{cosec}\theta)\,\mu^{-1}\hat{\boldsymbol{d}}_1, \\ 2W_2 &= \hat{\boldsymbol{d}}_2^{\mathrm{T}} \kappa^{-1} \hat{\boldsymbol{d}}_2, \\ 2\boldsymbol{R}_2 &= N_2 \operatorname{cosec}\theta(\hat{\boldsymbol{d}}_2^{\mathrm{T}} \kappa^{-1} \hat{\boldsymbol{d}}_2)\,\mu^{-1}\hat{\boldsymbol{d}}_1 \times \kappa^{-1}\hat{\boldsymbol{d}}_2. \end{aligned}\right\} \tag{7.10}$$

It is easy to check that $N_2 \boldsymbol{R}_2 \cdot \hat{\boldsymbol{n}} = W_2$, so confirming again (5.6).

Remark 2. *Energy density and energy flux for the superposition of two wave trains*

For given $\hat{\boldsymbol{n}}$ and ω, any linear combination

$$(\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}) = a(\boldsymbol{E}_1, \boldsymbol{H}_1, \boldsymbol{D}_1, \boldsymbol{B}_1) \exp \mathrm{i}\omega(N_1 \hat{\boldsymbol{n}} \cdot \boldsymbol{x} - t) + b(\boldsymbol{E}_2, \boldsymbol{H}_2, \boldsymbol{D}_2, \boldsymbol{B}_2) \exp \mathrm{i}\omega(N_2 \hat{\boldsymbol{n}} \cdot \boldsymbol{x} - t) \tag{7.11}$$

of the fields of the two wave trains (7.13) and (7.14), with slownesses $N_1 \hat{\boldsymbol{n}}$ and $N_2 \hat{\boldsymbol{n}}$ respectively, is also a solution of the field and constitutive equations (2.1)–(2.5). The fields (7.11) will be called the 'resultant fields'.

For the fields the mean energy flux and energy density are defined by (5.2) as in the case of a single wave train. The mean energy density $\tilde{\mathbf{W}}$ for the resultant fields is given by

$$\tilde{\mathbf{W}} = \tilde{\mathbf{W}}_1 + \tilde{\mathbf{W}}_2 = W_1 + W_2, \tag{7.12}$$

the sum of the weighted energy densities for the individual waves. The mean energy flux $\tilde{\mathbf{R}}$ for the resultant fields is given by

$$\tilde{\mathbf{R}} = \tilde{\mathbf{R}}_1 + \tilde{\mathbf{R}}_2 + \tilde{\mathbf{J}} = \boldsymbol{R}_1 + \boldsymbol{R}_2 + \tilde{\mathbf{J}}, \tag{7.13}$$

where $\tilde{\mathbf{J}}$, which is a function of $\boldsymbol{x}$, is given by

$$\tilde{\mathbf{J}} = \tfrac{1}{2}\{\bar{a}b \exp \mathrm{i}\omega(N_2 - N_1)\,\hat{\boldsymbol{n}} \cdot \boldsymbol{x}\}^{+} \boldsymbol{J}, \tag{7.14}$$

with

$$\boldsymbol{J} = \{N_1(\hat{\boldsymbol{d}}_1^{\mathrm{T}}\kappa^{-1}\hat{\boldsymbol{d}}_1)\,\kappa^{-1}\hat{\boldsymbol{d}}_2\times\mu^{-1}\hat{\boldsymbol{d}}_2 - N_2(\hat{\boldsymbol{d}}_2^{\mathrm{T}}\kappa^{-1}\hat{\boldsymbol{d}}_2)\,\kappa^{-1}\hat{\boldsymbol{d}}_1\times\mu^{-1}\hat{\boldsymbol{d}}_1\}\operatorname{cosec}\theta. \tag{7.15}$$

Now, because $\hat{\boldsymbol{d}}_1^{\mathrm{T}}\kappa^{-1}\hat{\boldsymbol{d}}_2 = \hat{\boldsymbol{d}}_1^{\mathrm{T}}\mu^{-1}\hat{\boldsymbol{d}}_2 = 0$, it follows that the vector $\kappa^{-1}\hat{\boldsymbol{d}}_2\times\mu^{-1}\hat{\boldsymbol{d}}_2$ is along $\hat{\boldsymbol{d}}_1$ and that the vector $\kappa^{-1}\hat{\boldsymbol{d}}_1\times\mu^{-1}\hat{\boldsymbol{d}}_1$ is along $\hat{\boldsymbol{d}}_2$. Thus $\tilde{\mathbf{J}}\cdot\hat{\boldsymbol{n}} = 0$ and we obtain

$$\tilde{\mathbf{R}}\cdot\hat{\boldsymbol{n}} = \boldsymbol{R}_1\cdot\hat{\boldsymbol{n}} + \boldsymbol{R}_2\cdot\hat{\boldsymbol{n}} = W_1 N_1^{-1} + W_2 N_2^{-1}, \tag{7.16}$$

confirming the result obtained by Hayes (1980, equation (6.27)).

8. Circularly polarized homogeneous waves: generalized optic axes

For magnetically isotropic, electrically anisotropic crystals, it was shown (Hayes 1987) that waves with both fields **D** and **B** circularly polarized are possible for homogeneous waves if and only if the secular equations has a double root. In this section we consider homogeneous waves and investigate when the secular equation (3.15) and (3.16) (with $\boldsymbol{C} = \hat{\boldsymbol{n}}$) has a double root.

In §8.1, it is shown that if for given $\hat{\boldsymbol{n}}$ the secular equation has a double root, then the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane $\Pi(\hat{\boldsymbol{n}})$, orthogonal to $\hat{\boldsymbol{n}}$, are similar and similarly situated. Next, in §8.2, it is shown conversely that if the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane $\Pi(\hat{\boldsymbol{n}})$ are similar, and similarly situated, then the secular equation has a double root.

In the case of a double root of the secular equation (3.15) (and (3.16)), the amplitude $\boldsymbol{D}$ ($\boldsymbol{B}$) may be any bivector in the plane $\Pi(\hat{\boldsymbol{n}})$ and owing to (4.2) the ellipse of $\boldsymbol{B}$ ($\boldsymbol{D}$) is similar, and similarly situated, to the polar reciprocal of the ellipse of $\boldsymbol{D}$ ($\boldsymbol{B}$) with respect to either of the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane $\Pi(\hat{\boldsymbol{n}})$. Also, because the double root N^{-2} is real and positive, N is real, and the real and imaginary parts of (3.1 *c*), (3.1 *d*) read

$$\boldsymbol{B}^{+} = N\hat{\boldsymbol{n}}\times\boldsymbol{E}^{+},\quad \boldsymbol{B}^{-} = N\hat{\boldsymbol{n}}\times\boldsymbol{E}^{-}, \tag{8.1}$$

$$\boldsymbol{D}^{+} = -N\hat{\boldsymbol{n}}\times\boldsymbol{H}^{+},\quad \boldsymbol{D}^{-} = -N\hat{\boldsymbol{n}}\times\boldsymbol{H}^{-}. \tag{8.2}$$

It follows obviously that

$$\boldsymbol{E}^{+}\cdot\boldsymbol{B}^{+} = \boldsymbol{E}^{-}\cdot\boldsymbol{B}^{-} = \boldsymbol{D}^{+}\cdot\boldsymbol{H}^{+} = \boldsymbol{D}^{-}\cdot\boldsymbol{H}^{-} = 0,\quad \boldsymbol{D}^{+}\cdot\boldsymbol{E}^{+} = \boldsymbol{B}^{+}\cdot\boldsymbol{H}^{+}, \boldsymbol{D}^{-}\cdot\boldsymbol{E}^{-} = \boldsymbol{B}^{-}\cdot\boldsymbol{H}^{-}, \tag{8.3}$$

so that

$$\boldsymbol{D}^{+\mathrm{T}}\kappa^{-1}\boldsymbol{B}^{+} = \boldsymbol{D}^{+\mathrm{T}}\mu^{-1}\boldsymbol{B}^{+} = 0,\quad \boldsymbol{D}^{+\mathrm{T}}\kappa^{-1}\boldsymbol{D}^{+} = \boldsymbol{B}^{+\mathrm{T}}\mu^{-1}\boldsymbol{B}^{+}, \tag{8.4}$$

and

$$\boldsymbol{D}^{-\mathrm{T}}\kappa^{-1}\boldsymbol{B}^{-} = \boldsymbol{D}^{-\mathrm{T}}\mu^{-1}\boldsymbol{B}^{-} = 0,\quad \boldsymbol{D}^{-\mathrm{T}}\kappa^{-1}\boldsymbol{D}^{-} = \boldsymbol{B}^{-\mathrm{T}}\mu^{-1}\boldsymbol{B}^{-}. \tag{8.5}$$

Thus, $\boldsymbol{D}^{+}$ and $\boldsymbol{B}^{+}$ are conjugate with respect to either of the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane $\Pi(\hat{\boldsymbol{n}})$, and so are $\boldsymbol{D}^{-}$ and $\boldsymbol{B}^{-}$. Also, if $\boldsymbol{D}^{+}$ ($\boldsymbol{D}^{-}$) is a radius to the κ^{-1}-metric ellipsoid, then $\boldsymbol{B}^{+}$ ($\boldsymbol{B}^{-}$) is a radius to the μ^{-1}-metric ellipsoid. The contrast between this and the classical case (μ isotropic) is illustrated in figure 2.

Next, in §8.3, we seek the directions $\hat{\boldsymbol{n}}$ for which the sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane $\Pi(\hat{\boldsymbol{n}})$ are similar and similarly situated. We call these directions 'generalized optic axes'. We find that, for given κ and μ, there is either one such direction (uniaxial case), two such directions (biaxial case), or an infinity (pseudo-isotropic case).

For $\hat{\boldsymbol{n}}$ along a generalized optic axis, $\boldsymbol{D}$ ($\boldsymbol{B}$) may in particular be chosen to be isotropic in the plane $\Pi(\hat{\boldsymbol{n}})$ orthogonal to $\hat{\boldsymbol{n}}$, and then the ellipse of $\boldsymbol{B}$ ($\boldsymbol{D}$) is the polar reciprocal of the circle of $\boldsymbol{D}$ ($\boldsymbol{B}$) with respect to either of the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by

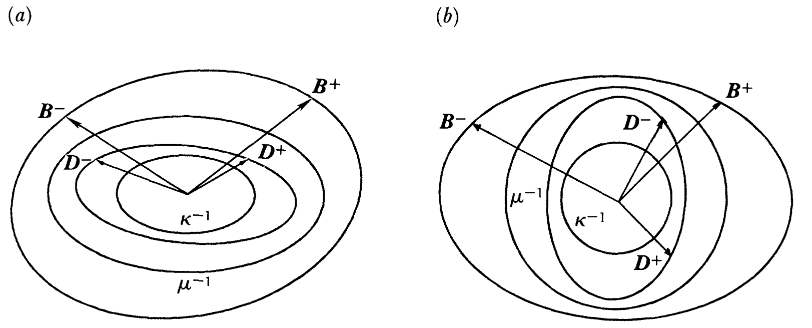

FIGURE 2. Propagation of elliptically polarized homogeneous waves along a (generalized) optic axis. In (*a*), both μ and κ are anisotropic: the sections of the μ^{-1} and κ^{-1}-metric ellipsoids by $\Pi(\hat{\boldsymbol{n}})$ are similar and similarly situated ellipses; $\boldsymbol{D}^{+}$ ($\boldsymbol{D}^{-}$) and $\boldsymbol{B}^{+}$ ($\boldsymbol{B}^{-}$) are along conjugate directions with respect to these ellipses. In (*b*), μ is isotropic: the sections of the μ^{-1} and κ^{-1}-metric ellipsoids by $\Pi(\hat{\boldsymbol{n}})$ are both circles; $\boldsymbol{D}^{+}$ ($\boldsymbol{D}^{-}$) is orthogonal to $\boldsymbol{B}^{+}$ ($\boldsymbol{B}^{-}$).

the plane $\Pi(\hat{\boldsymbol{n}})$. In this case the **D** (**B**) field is circularly polarized while the **B** (**D**) field is elliptically polarized.

Again, in particular, the ellipse of $\boldsymbol{D}$ ($\boldsymbol{B}$) may be chosen to be similar and similarly situated to either of the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids. Then it follows (Remark 1 of Appendix B) that the ellipse of $\boldsymbol{B}$ ($\boldsymbol{D}$) is similar and similarly situated to the same elliptical sections. In this case $\boldsymbol{D}$ and $\boldsymbol{B}$ are parallel bivectors and both the fields **D** and **B** are elliptically polarized. Also, in this case $\boldsymbol{D}^{+}$ and $\boldsymbol{D}^{-}$ are along conjugate diameters of the same elliptical sections, and so are $\boldsymbol{B}^{+}$ and $\boldsymbol{B}^{-}$. With (8.4) and (8.5) this implies that $\boldsymbol{D}^{+}$ is parallel to $\boldsymbol{B}^{-}$ and $\boldsymbol{D}^{-}$ is parallel to $\boldsymbol{B}^{+}$. This is illustrated in figure 3*c*, *d*, which presents remarkable special cases of figure 2. Note that in figure 3*c* the ellipse of $\boldsymbol{D}$ has been chosen identical with the elliptical section of the κ^{-1}-metric ellipsoid by the plane $\Pi(\hat{\boldsymbol{n}})$. The type of wave described by figure 3*c* may be regarded as a generalization of the classical circularly polarized wave.

Also, in particular, $\boldsymbol{D}$ ($\boldsymbol{B}$) may be chosen to be any real vector in the plane $\Pi(\hat{\boldsymbol{n}})$ orthogonal to $\hat{\boldsymbol{n}}$, and the $\boldsymbol{B}$ ($\boldsymbol{D}$) is also a real vector, and both are conjugate with respect to either of the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane $\Pi(\hat{\boldsymbol{n}})$. The wave is then linearly polarized. This is also illustrated in figure 3 (*a* and *b*). Note that, in figure 3*a*, $\boldsymbol{D}$ has been chosen to be a radius of the κ^{-1}-metric ellipsoid.

Finally, in §8.4 we present analytical details for homogeneous wave propagation along a generalized optic axis. The solutions are written as linear combinations of two linearly polarized waves. Of course these waves may be combined to give two waves polarized as shown on figure 3*c* and of opposite handedness.

8.1. *Double roots*

Let us first assume that the secular equation (3.15) (and (3.16)) with $\boldsymbol{C} = \hat{\boldsymbol{n}}$ has a double root. This root is then a double eigenvalue of the eigenvalue problem (3.4) (and (3.5)) for the real symmetric matrix $\Gamma\kappa^{-1}\Gamma$ ($\Gamma\mu^{-1}\Gamma$) with respect to the metric κ (μ). Then, to this eigenvalue correspond two linearly independent real eigenvectors $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, ($\boldsymbol{H}_1$, $\boldsymbol{H}_2$) and any linear combination of them with real or complex coefficients is an eigenvector or eigenbivector corresponding to this same eigenvalue (see, for instance, Goldstein 1950, ch. X).

From the constitutive equations (3.2), $\boldsymbol{D}_1 = \kappa\boldsymbol{E}_1$, $\boldsymbol{D}_2 = \kappa\boldsymbol{E}_2$, ($\boldsymbol{B}_1 = \mu\boldsymbol{H}_1$, $\boldsymbol{B}_2 = \mu\boldsymbol{H}_2$) are two linearly independent real vectors. From (3.1*a*, *b*) these are in the plane $\Pi(\hat{\boldsymbol{n}})$, so that $\boldsymbol{D}$ ($\boldsymbol{B}$)

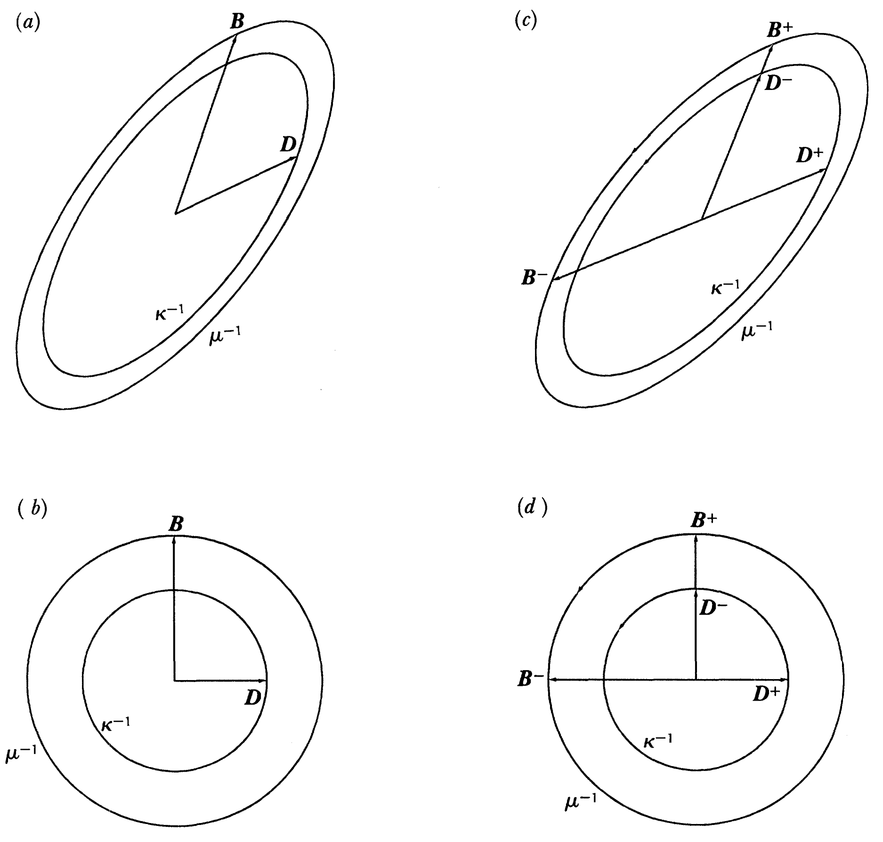

FIGURE 3. Propagation along a (generalized) optic axis (special cases). Figures (*a*) and (*b*) are special cases of figure 2*a*. In (*a*) the wave is linearly polarized: $\boldsymbol{D}$ and $\boldsymbol{B}$ are conjugate radii of the similar and similarly situated elliptical sections of the μ^{-1} and κ^{-1}-metric ellipsoids by $\Pi(\hat{\boldsymbol{n}})$. In (*c*) the wave is elliptically polarized: the ellipses of $\boldsymbol{D}$ and $\boldsymbol{B}$ are both similar, and similarly situated, to the elliptical sections of the μ^{-1} and κ^{-1}-metric ellipsoids by $\Pi(\hat{\boldsymbol{n}})$. Figures (*b*) and (*d*) are special cases of figure 2*b*. In (*b*) the wave is linearly polarized with $\boldsymbol{D}$ orthogonal to $\boldsymbol{B}$. In (*d*) the wave is circularly polarized with $\boldsymbol{B}^{+}$ ($\boldsymbol{D}^{-}$) orthogonal to $\boldsymbol{D}^{+}$ ($\boldsymbol{B}^{-}$).

corresponding to a double root of the secular equation may be any real vector or bivector in this plane. Recalling (4.2), the real vectors $\boldsymbol{D}$ and $\boldsymbol{B}$ must be along common conjugate directions of the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane $\Pi(\hat{\boldsymbol{n}})$. It follows that in the case of a double root for the secular equation these elliptical sections must have an infinity of common conjugate directions. This means that these elliptical sections are similar, and similarly situated.

Remark

We present an alternative proof based on the results of §7. Let $\hat{\boldsymbol{d}}_1$ and $\hat{\boldsymbol{d}}_2$ be unit vectors along common conjugate directions of the elliptical sections of the κ^{-1}- and μ^{-1}-metric ellipsoids by the plane $\Pi(\hat{\boldsymbol{n}})$. The lengths a_1 and a_2 of the conjugate radii along $\hat{\boldsymbol{d}}_1$ and $\hat{\boldsymbol{d}}_2$, to the κ^{-1}-metric ellipsoid, and the lengths b_1 and b_2 of the conjugate radii along $\hat{\boldsymbol{d}}_1$ and $\hat{\boldsymbol{d}}_2$, to the μ^{-1}-metric ellipsoid, are given by

$$a_\alpha^2 = (\boldsymbol{d}_\alpha^{\mathrm{T}} \kappa^{-1} \hat{\boldsymbol{d}}_\alpha)^{-1}, \quad b_\alpha^2 = (\boldsymbol{d}_\alpha^{\mathrm{T}} \mu^{-1} \hat{\boldsymbol{d}}_\alpha)^{-1}, \quad \alpha = 1, 2 \quad \text{(no sum)}.$$

The slownesses N_1 and N_2, of the waves with amplitude $\boldsymbol{D}$ along $\hat{\boldsymbol{d}}_1$ and $\hat{\boldsymbol{d}}_2$ respectively, are given by the equations (7.8). Now, if $N_1 = N_2$, it follows from (7.8) that $a_1 b_2 = a_2 b_1$, that is $b_1/a_1 = b_2/a_2$. This expresses the fact that the lengths of the conjugate radii (along $\hat{\boldsymbol{d}}_1, \hat{\boldsymbol{d}}_2$) of the elliptical section of the μ^{-1}-metric ellipsoid are proportional to those of the elliptical section of the κ^{-1}-metric ellipsoid. Thus the two ellipses are similar, and similarly situated.

8.2. *Similar and similarly situated elliptical sections*

Conversely, let us now assume that $\hat{\boldsymbol{n}}$ is such that the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane $\Pi(\hat{\boldsymbol{n}})$ orthogonal to $\hat{\boldsymbol{n}}$ are similar, and similarly situated. We have then for some real scalar $\lambda \neq 0$,

$$\Gamma\mu^{-1}\Gamma = \lambda\Gamma\kappa^{-1}\Gamma. \tag{8.6}$$

To see this suppose the x_3-axis is chosen along $\hat{\boldsymbol{n}}$. Then $\hat{n}_i = \delta_{i3}$, and the only non-zero components of Γ are $\Gamma_{12} = -1 = -\Gamma_{21}$. Then equation (8.6) expresses the fact that

$$\mu_{11}^{-1}/\kappa_{11}^{-1} = \mu_{22}^{-1}/\kappa_{22}^{-1} = \mu_{12}^{-1}/\kappa_{12}^{-1} = \lambda, \tag{8.7}$$

so that λ is the similarity factor that transforms the elliptical section of the κ^{-1}-metric ellipsoid by the plane $\Pi(\hat{\boldsymbol{n}})$ into that of the μ^{-1}-ellipsoid.

Then the two equivalent forms (3.12) and (3.13) of the secular equation reduce to

$$\det(\lambda\Gamma\kappa^{-1}\Gamma + N^{-2}\kappa) = 0, \quad \det(\lambda^{-1}\Gamma\mu^{-1}\Gamma + N^{-2}\mu) = 0. \tag{8.8}$$

Thus, in the explicit form (3.15), μ^{-1} may now be replaced by κ^{-1} and in the explicit form (3.16), κ^{-1} may now be replaced by μ^{-1}. This leads to

$$(\lambda\hat{\boldsymbol{n}}^{\mathrm{T}}\kappa\hat{\boldsymbol{n}} - N^{-2}\det\kappa)^2 = 0, \quad (\lambda^{-1}\hat{\boldsymbol{n}}^{\mathrm{T}}\mu\hat{\boldsymbol{n}} - N^{-2}\det\mu)^2 = 0, \tag{8.9}$$

which shows that the secular equation has the double root

$$N^{-2} = \lambda(\det\kappa)^{-1}\hat{\boldsymbol{n}}^{\mathrm{T}}\kappa\hat{\boldsymbol{n}} = \lambda^{-1}(\det\mu)^{-1}\hat{\boldsymbol{n}}^{\mathrm{T}}\mu\hat{\boldsymbol{n}}. \tag{8.10}$$

8.3. *Generalized optic axes*

We now seek the directions $\hat{\boldsymbol{n}}$ such that the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane $\Pi(\hat{\boldsymbol{n}})$ orthogonal to $\hat{\boldsymbol{n}}$ are similar, and similarly situated. This problem is a generalization of the problem of finding the circular sections of a given ellipsoid. Indeed, one knows (Bocher 1907) that there exists a real non singular matrix T that diagonalizes simultaneously the two real symmetric positive definite matrices κ^{-1}, μ^{-1}:

$$T^{\mathrm{T}}\mu^{-1}T = \begin{pmatrix} m_1^{-1} & 0 & 0 \\ 0 & m_2^{-1} & 0 \\ 0 & 0 & m_3^{-1} \end{pmatrix}, \quad T^{\mathrm{T}}\kappa^{-1}T = \begin{pmatrix} k_1^{-1} & 0 & 0 \\ 0 & k_2^{-1} & 0 \\ 0 & 0 & k_3^{-1} \end{pmatrix}. \tag{8.11}$$

To obtain the matrix T, it is necessary to solve the eigenvalue problem for the matrix κ^{-1} (μ^{-1}) with respect to the metric $\mu^{-1}(\kappa^{-1})$, that is

$$(\mu^{-1} - \lambda\kappa^{-1})\,V = 0. \tag{8.12}$$

The columns of the matrix T are three independent eigenvectors $(\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3)$ orthogonal with respect to the κ^{-1} and μ^{-1}-matrices:

$$T = (\boldsymbol{V}_1|\boldsymbol{V}_2|\boldsymbol{V}_3), \quad \boldsymbol{V}_i^{\mathrm{T}}\kappa^{-1}\boldsymbol{V}_j = \boldsymbol{V}_i^{\mathrm{T}}\mu^{-1}\boldsymbol{V}_j = 0 \quad (i \neq j), \tag{8.13}$$

and the real positive eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are the ratios

$$\lambda_1 = \frac{m_1^{-1}}{k_1^{-1}} = \frac{\boldsymbol{V}_1^{\mathrm{T}}\mu^{-1}\boldsymbol{V}_1}{\boldsymbol{V}_1^{\mathrm{T}}\kappa^{-1}\boldsymbol{V}_1}, \quad \lambda_2 = \frac{m_2^{-1}}{k_2^{-1}} = \frac{\boldsymbol{V}_2^{\mathrm{T}}\mu^{-1}\boldsymbol{V}_2}{\boldsymbol{V}_2^{\mathrm{T}}\kappa^{-1}\boldsymbol{V}_2}, \quad \lambda_3 = \frac{m_3^{-1}}{k_3^{-1}} = \frac{\boldsymbol{V}_3^{\mathrm{T}}\mu^{-1}\boldsymbol{V}_3}{\boldsymbol{V}_3^{\mathrm{T}}\kappa^{-1}\boldsymbol{V}_3}. \tag{8.14}$$

For the sake of simplicity, the eigenvectors $\boldsymbol{V}_1$, $\boldsymbol{V}_2$, $\boldsymbol{V}_3$ may be chosen to be equal in magnitude, and such that $\det T = 1$. Then, from (8.11),

$$\det\mu^{-1} = (m_1 m_2 m_3)^{-1}, \quad \det\kappa^{-1} = (k_1 k_2 k_3)^{-1}. \tag{8.15}$$

In the case of magnetically isotropic crystals, the vectors $\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3$ are then unit vectors along the principal axes of the κ^{-1}-metric ellipsoid (dielectric ellipsoid) or of the κ-metric ellipsoid (Fresnel ellipsoid). They are of course orthogonal. However, in the present case, when μ and κ do not have the same principal axes, the vectors $\boldsymbol{V}_1$, $\boldsymbol{V}_2$, $\boldsymbol{V}_3$ lie along oblique axes.

It is convenient to introduce the reciprocal set $\boldsymbol{V}_*^1$, $\boldsymbol{V}_*^2$, $\boldsymbol{V}_*^3$ of the set $\boldsymbol{V}_1$, $\boldsymbol{V}_2$, $\boldsymbol{V}_3$, defined by

$$\boldsymbol{V}_*^i \cdot \boldsymbol{V}_j = \delta_j^i, \quad \text{or} \quad \boldsymbol{V}_*^i = k_i \kappa^{-1}\boldsymbol{V}_i = m_i \mu^{-1}\boldsymbol{V}_i \quad (\text{no sum}), \tag{8.16}$$

because $\boldsymbol{V}_*^i$ must satisfy $(\mu - \lambda_i^{-1}\kappa)\, \boldsymbol{V}_*^i = 0$ (no sum). Alternatively

$$\boldsymbol{V}_*^i = \tfrac{1}{2}\epsilon^{ijk}\boldsymbol{V}_j \times \boldsymbol{V}_k, \quad \boldsymbol{V}_i = \tfrac{1}{2}\epsilon_{ijk}\boldsymbol{V}_*^j \times \boldsymbol{V}_*^k, \tag{8.17}$$

on using equation (8.15). We note that $\boldsymbol{V}_i$ and $\boldsymbol{V}_*^i$ are orthogonal and identical unit vectors if and only if the tensor κ has the same principal axes as the tensor μ. This case arises in photoelasticity (Smith & Rivlin 1970).

The matrix T_*, defined by

$$T_* = (\boldsymbol{V}_*^1 | \boldsymbol{V}_*^2 | \boldsymbol{V}_*^3), \tag{8.18}$$

is such that

$$T^{\mathrm{T}} T_* = 1 = T_*^{\mathrm{T}} T, \tag{8.19}$$

and diagonalizes simultaneously the matrices κ and μ:

$$T_*^{\mathrm{T}} \mu T_* = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \quad T_*^{\mathrm{T}} \kappa T_* = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}. \tag{8.20}$$

Three cases have to be considered: either the three eigenvalues (8.14) are all different, or two of them are equal, or all three are equal.

Case 1. *Biaxial crystal*: $\lambda_1 > \lambda_2 > \lambda_3$

In this case there are two and only two planes Π_+ and Π_- (say) such that for each plane its intersections with the κ^{-1} and μ^{-1}-metric ellipsoids are similar and similarly situated ellipses.

The plane Π_+ is spanned by the vector $\boldsymbol{V}_2$ (along the 'intermediate axis') and the vector $\boldsymbol{V}_+$ defined by

$$\boldsymbol{V}_+ = \gamma \boldsymbol{V}_1 + \alpha \boldsymbol{V}_3, \tag{8.21}$$

and the plane Π_- is spanned by the vector $\boldsymbol{V}_2$ and the vector $\boldsymbol{V}_-$ defined by

$$\boldsymbol{V}_- = \gamma \boldsymbol{V}_1 - \alpha \boldsymbol{V}_3, \tag{8.22}$$

with γ and α given by

$$\left.\begin{aligned}\gamma &= \left(\frac{k_1(\lambda_2-\lambda_3)}{k_2(\lambda_1-\lambda_3)}\right)^{\frac{1}{2}} = \left(\frac{m_1(\lambda_3^{-1}-\lambda_2^{-1})}{m_2(\lambda_3^{-1}-\lambda_1^{-1})}\right)^{\frac{1}{2}},\\ \alpha &= \left(\frac{k_3(\lambda_1-\lambda_2)}{k_2(\lambda_1-\lambda_3)}\right)^{\frac{1}{2}} = \left(\frac{m_3(\lambda_2^{-1}-\lambda_1^{-1})}{m_2(\lambda_3^{-1}-\lambda_1^{-1})}\right)^{\frac{1}{2}}.\end{aligned}\right\} \tag{8.23}$$

We note the identities

$$m_3^{-1}\alpha^2+m_1^{-1}\gamma^2 = m_2^{-1}, \quad k_3^{-1}\alpha^2+k_1^{-1}\gamma^2 = k_2^{-1}. \tag{8.24}$$

For future reference we introduce here the bivectors $\boldsymbol{A}_+$, $\boldsymbol{A}_-$, defined by

$$\boldsymbol{A}_\pm = \pm\gamma\boldsymbol{V}_1+\mathrm{i}\boldsymbol{V}_2+\alpha\boldsymbol{V}_3. \tag{8.25}$$

Then, the ellipse of the bivector $\boldsymbol{A}_+$ ($\boldsymbol{A}_-$) or of its complex conjugate is similar, and similarly situated, to either of the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane Π_+ (Π_-).

The directions of the generalized optic axes are along normals $\boldsymbol{n}_\pm$ (say) to the planes $\Pi_\pm$ of the bivectors $\boldsymbol{A}_\pm$ and are thus given by the cross products $\boldsymbol{V}_2\times\boldsymbol{A}_\pm$. Indeed, by using (8.17),

$$\boldsymbol{n}_\pm = \pm\alpha\boldsymbol{V}_*^1-\gamma\boldsymbol{V}_*^3. \tag{8.26}$$

These are the only two directions in which homogeneous waves with either the field **D** or the field **B** circularly polarized may propagate.

In the special case when the crystal is magnetically isotropic, that is $\mu_{ij} = \mu\delta_{ij}$, (8.23) reduces to

$$\gamma = \left(\frac{k_1(k_2-k_3)}{k_2(k_1-k_3)}\right)^{\frac{1}{2}}, \quad \alpha = \left(\frac{k_3(k_1-k_2)}{k_2(k_1-k_3)}\right)^{\frac{1}{2}}, \tag{8.27}$$

where $k_1 > k_2 > k_3$ are now the principal values of the tensor κ. In this case the vectors $\boldsymbol{n}_\pm$ given by (8.26) are along the well-known optic axes (Born & Wolf 1980) of the biaxial crystal, because $\boldsymbol{V}_*^1$, $\boldsymbol{V}_*^2$, $\boldsymbol{V}_*^3$ are now unit vectors in the principal directions of the tensor κ.

Case 2. *Uniaxial crystals*: $\lambda_1 = \lambda_2 > \lambda_3$ *or* $\lambda_1 > \lambda_2 = \lambda_3$.

In this case there is one and only one plane Π_0 (say) such that its intersections with the κ^{-1} and μ^{-1}-metric ellipsoids are similar, and similarly situated, ellipses.

The plane Π_0 is spanned by the vectors $\boldsymbol{V}_1$, $\boldsymbol{V}_2$ when $\lambda_1 = \lambda_2$, and by the vectors $\boldsymbol{V}_2$, $\boldsymbol{V}_3$ when $\lambda_2 = \lambda_3$.

For the purpose of further reference we introduce here the bivector $\boldsymbol{A}_0$ defined by

$$\boldsymbol{A}_0 = (k_1/k_2)^{\frac{1}{2}}\boldsymbol{V}_1+\mathrm{i}\boldsymbol{V}_2 = (m_1/m_2)^{\frac{1}{2}}\boldsymbol{V}_1+\mathrm{i}\boldsymbol{V}_2, \tag{8.28}$$

or

$$\boldsymbol{A}_0 = \mathrm{i}\boldsymbol{V}_2+(k_3/k_2)^{\frac{1}{2}}\boldsymbol{V}_3 = \mathrm{i}\boldsymbol{V}_2+(m_3/m_2)^{\frac{1}{2}}\boldsymbol{V}_3, \tag{8.29}$$

respectively for the case when $\lambda_1 = \lambda_2$ or the case when $\lambda_2 = \lambda_3$. Then the ellipse of the bivector $\boldsymbol{A}_0$ or of its complex conjugate is similar and similarly situated to either of the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane Π_0.

The optic axis is along the normal $\boldsymbol{n}_0$ (say) to the plane Π_0 of the bivector $\boldsymbol{A}_0$:

$$\boldsymbol{n}_0 = \boldsymbol{V}_*^3, \tag{8.30}$$

or

$$\boldsymbol{n}_0 = \boldsymbol{V}_*^1, \tag{8.31}$$

respectively for the case when $\lambda_1 = \lambda_2$ or when $\lambda_2 = \lambda_3$. This is the only direction in which homogeneous waves with either the field **D** or the field **B** circularly polarized may propagate.

In the special case when the crystal is magnetically isotropic, that is $\mu_{ij} = \mu\delta_{ij}$, we have $k_1 = k_2 > k_3$ or $k_1 > k_2 = k_3$ where k_1, k_2, k_3 are now the principal values of κ, and $\boldsymbol{n}_0$ is respectively along the third or first principal direction of κ. This is the well-known optic axis (Born & Wolf 1980) of the uniaxial case.

Case 3. *Pseudo-isotropic crystal*: $\lambda_1 = \lambda_2 = \lambda_3$

In this case the κ^{-1}- and μ^{-1}-metric ellipsoids are similar, and similarly situated, so that the elliptical sections of these ellipsoids by any plane are also similar and similarly situated. In this case, every direction is a generalized optic axis, and homogeneous waves with either the field **D** or the field **B** circularly polarized may propagate in all directions, as is the case for isotropic media. However, as may be seen from (8.10), the phase speed (or the refractive index) of these waves in general depends on the propagation direction $\hat{\boldsymbol{n}}$, which shows that the crystal is not isotropic. In the special case when the crystal is magnetically isotropic, that is $\mu_{ij} = \mu\delta_{ij}$, we have also $\kappa_{ij} = \kappa\delta_{ij}$, and the crystal is isotropic.

8.4. *Analytical results for wave propagation along an optic axis*

Case 1. *Biaxial crystal*: $\lambda_1 > \lambda_2 > \lambda_3$

Let us consider the propagation of homogeneous waves along the direction of the generalized optic axis $\boldsymbol{n}_+ = \alpha \boldsymbol{V}^1_* - \gamma \boldsymbol{V}^3_*$. The corresponding double root N^{-2} of the secular equation is given by (8.10) with $\boldsymbol{n} = \boldsymbol{n}_+$. Using the identities (8.24) we obtain

$$N^{-2} = (k_2 m_2)^{-1}. \tag{8.32}$$

As the bivector $\boldsymbol{D}$ may be any bivector in the plane orthogonal to $\boldsymbol{n}_+$, it may be written as a linear combination with arbitrary coefficients of the vectors $\boldsymbol{V}_2$ and $\alpha\boldsymbol{V}_1 + \gamma\boldsymbol{V}_3$. Thus, using (3.1), (3.2) and (8.24) we get

$$\left.\begin{aligned} \boldsymbol{D} &= ak_2^{\frac{1}{2}}\boldsymbol{V}_2 + bk_2^{\frac{1}{2}}(\gamma\boldsymbol{V}_1 + \alpha\boldsymbol{V}_3), \\ \boldsymbol{E} &= ak_2^{-\frac{1}{2}}\boldsymbol{V}^2_* + bk_2^{\frac{1}{2}}(\gamma k_1^{-1}\boldsymbol{V}^1_* + \alpha k_3^{-1}\boldsymbol{V}^3_*), \\ \boldsymbol{B} &= -bm_2^{\frac{1}{2}}\boldsymbol{V}_2 + am_2^{\frac{1}{2}}(\gamma\boldsymbol{V}_1 + \alpha\boldsymbol{V}_3), \\ \boldsymbol{H} &= -bm_2^{-\frac{1}{2}}\boldsymbol{V}^2_* + am_2^{\frac{1}{2}}(\gamma m_1^{-1}\boldsymbol{V}^1_* + \alpha m_3^{-1}\boldsymbol{V}^3_*), \end{aligned}\right\} \tag{8.33}$$

where a and b are arbitrary, possibly complex, coefficients.

We note that for $a = \pm ib$ we obtain two waves with the bivectors $\boldsymbol{D}$ and $\boldsymbol{B}$ parallel. With the plus sign, $\boldsymbol{D}$ and $\boldsymbol{B}$ are parallel to the bivector $\boldsymbol{A}_+$ defined by (8.25), and with the minus sign they are parallel to its complex conjugate $\bar{\boldsymbol{A}}_+$. These two waves are of the type described by figure 3 and are polarized with opposite handedness.

Also denoting by $\boldsymbol{D}_1$, $\boldsymbol{E}_1$, $\boldsymbol{B}_1$, $\boldsymbol{H}_1$ the fields of the wave corresponding to $a = 1$, $b = 0$ and by $\boldsymbol{D}_2$, $\boldsymbol{E}_2$, $\boldsymbol{B}_2$, $\boldsymbol{H}_2$ the fields of the wave corresponding to $a = 0$, $b = 1$, we note that the orthogonality relations of §4.2 are satisfied by these two linearly polarized waves. The corresponding weighted energy densities and energy fluxes are given by

$$W_1 = \tfrac{1}{2}, \quad \boldsymbol{R}_1 = \tfrac{1}{2}(m_1 m_3)^{-1}\lambda_2^{-\frac{1}{2}}(\alpha m_1\boldsymbol{V}_1 - \gamma m_3\boldsymbol{V}_3) = \tfrac{1}{2}(m_1 m_2)^{-1}\lambda_2^{-\frac{1}{2}}\mu\boldsymbol{n}_+, \tag{8.34}$$

and

$$W_2 = \tfrac{1}{2}, \quad \boldsymbol{R}_2 = \tfrac{1}{2}(k_1 k_3)^{-1}\lambda_2^{\frac{1}{2}}(\alpha k_1 \boldsymbol{V}_1 - \gamma k_3 \boldsymbol{V}_3) = \tfrac{1}{2}(k_1 k_3)^{-1}\lambda_2^{\frac{1}{2}}\kappa \boldsymbol{n}_+. \tag{8.35}$$

For the combined wave (8.35) we obtain

$$W = \tfrac{1}{2}(|a|^2 + |b|^2), \tag{8.36}$$

in accordance with (5.22), and

$$\boldsymbol{R} = |a|^2 \boldsymbol{R}_1 + |b|^2 \boldsymbol{R}_2 - \tfrac{1}{2}(a\bar{b})^+ (k_2 m_2)^{\frac{1}{2}} (k_1 k_3)^{-1} \alpha\gamma(\lambda_1 - \lambda_3)\, \boldsymbol{V}_2. \tag{8.37}$$

The interaction term in $\boldsymbol{R}$ is orthogonal to the propagation direction $\boldsymbol{n}_+$ in accordance with (5.26).

Taking oblique axes along $\boldsymbol{R}_1$, $\boldsymbol{R}_2$ and $\boldsymbol{V}_2$, we note that for a and b real, the square of the component of $\boldsymbol{R}$ along $\boldsymbol{V}_2$ is equal to the product of the components along $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ multiplied by a constant. For a and b complex, the square of the component of $\boldsymbol{R}$ along $\boldsymbol{V}_2$ may take any value less than the product of the components along $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ multiplied by this constant. Thus for a and b real the corresponding energy fluxes $\boldsymbol{R}$ lie on an elliptical cone, while for a and b complex $\boldsymbol{R}$ may be any vector inside this cone. The vector $\boldsymbol{R}_1 + \boldsymbol{R}_2$ is inside this cone and the vectors $\boldsymbol{R}_1 - \boldsymbol{R}_2$ and $\boldsymbol{V}_2$ are parallel to conjugate directions of any elliptical section of this cone by a plane conjugate to the direction of $\boldsymbol{R}_1 + \boldsymbol{R}_2$.

Case 2. *Uniaxial crystal*: $\lambda_1 > \lambda_2 = \lambda_3$

Let us consider the propagation of homogeneous waves along the direction of the generalized optic axis $\boldsymbol{n}_0 = \boldsymbol{V}_*^1$. The corresponding double root N^{-2} of the secular equation, given by (8.10) is

$$N^{-2} = (k_2 m_3)^{-1} = (k_3 m_2)^{-1}. \tag{8.38}$$

As the bivector $\boldsymbol{D}$ may be any bivector in the plane orthogonal to $\boldsymbol{n}_0 = \boldsymbol{V}_*^1$, we obtain, using (3.1) and (3.2),

$$\left.\begin{aligned} \boldsymbol{D} &= a k_2^{\frac{1}{2}} \boldsymbol{V}_2 + b k_3^{\frac{1}{2}} \boldsymbol{V}_3, \\ \boldsymbol{E} &= a k_2^{-\frac{1}{2}} \boldsymbol{V}_*^2 + b k_3^{-\frac{1}{2}} \boldsymbol{V}_*^3, \\ \boldsymbol{B} &= -b m_2^{\frac{1}{2}} \boldsymbol{V}_2 + a m_3^{\frac{1}{2}} \boldsymbol{V}_3, \\ \boldsymbol{H} &= -b m_2^{-\frac{1}{2}} \boldsymbol{V}_*^2 + a m_3^{-\frac{1}{2}} \boldsymbol{V}_*^3, \end{aligned}\right\} \tag{8.39}$$

where a and b are arbitrary possibly complex coefficients.

We note that for $a = \pm ib$ we obtain two waves with the bivectors $\boldsymbol{D}$ and $\boldsymbol{B}$ parallel. With the plus sign, $\boldsymbol{D}$ and $\boldsymbol{B}$ are parallel to the bivector $\boldsymbol{A}_0$ defined by (8.29), and with the minus sign they are parallel to its complex conjugate $\bar{\boldsymbol{A}}_0$. These two waves are of the type described by figure 3*c*, *d* and are polarized with opposite handedness.

The weighted energy density and energy flux of the combined wave (8.39) are given by

$$W = \tfrac{1}{2}(|a|^2 + |b|^2), \tag{8.40}$$

$$\boldsymbol{R} = \tfrac{1}{2}(k_2 m_3)^{-\frac{1}{2}}(|a|^2 + |b|^2)\, \boldsymbol{V}_1 = N^{-1} W \boldsymbol{V}_1. \tag{8.41}$$

Thus in this case there is no interaction term in the weighted energy flux of the combined wave, which always lies along $\boldsymbol{V}_1$.

Case 3. *Pseudo-isotropic crystal*: $\lambda_1 = \lambda_2 = \lambda_3$

Any direction is a generalized optic axis. To obtain analytical results for the propagation along the direction $\boldsymbol{n}$, we here may choose $\boldsymbol{V}_*^1 = \boldsymbol{n}$. Then $\boldsymbol{V}_*^2$ and $\boldsymbol{V}_*^3$ may be chosen along any

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pair of conjugate directions of either of the elliptical sections of the κ^{-1}- and μ^{-1}-metric ellipsoids by the plane conjugate to $\boldsymbol{n}$ with respect to either of these ellipsoids. With such a choice of $\boldsymbol{V}^1_*$, $\boldsymbol{V}^2_*$, $\boldsymbol{V}^3_*$ the results are given by (8.38)–(8.41).

9. Biaxial crystals

In this section, the propagation of inhomogeneous plane waves in biaxial crystals is investigated in detail.

First, in §9.1, explicit forms of the secular equation and of the propagation conditions are derived referring $\boldsymbol{C}$, $\boldsymbol{E}$ and $\boldsymbol{H}$ to the basis $\boldsymbol{V}^i_*$.

In §9.2, the possibility of the secular equation (3.15) or (3.16) for N^{-2} having zero roots is considered. It is shown that this equation has a double zero root (no wave propagation) when the ellipse of $\boldsymbol{C}$ is similar and similarly situated to one of the elliptical sections of the κ- or μ-metric ellipsoid by the planes conjugate to the generalized optic axes with respect to this ellipsoid. There are four such elliptical sections (two of the κ-metric ellipsoid, and two of the μ-metric ellipsoid). We call them critical sections. No wave propagation is possible with a slowness bivector whose ellipse is similar, and similarly situated, to a critical section. Next it is shown that the secular equation has only one zero root (one propagation mode) when the ellipse of $\boldsymbol{C}$ is similar, and similarly situated, to an elliptical section of the κ- or μ-metric ellipsoid other than a critical section.

Next, in §9.3, the condition for a double (non zero) root N^{-2} of the secular equation (3.15) or (3.16) is written down. It turns out that this condition may be factored. It is then shown that corresponding to the double root there is a wave for which the bivectors $\boldsymbol{D}$ and $\boldsymbol{B}$ are parallel. The ellipses of $\boldsymbol{D}$ and $\boldsymbol{B}$ are then both similar, and similarly situated, to either of the elliptical sections of the κ^{-1}- and μ^{-1}-metric ellipsoids by the plane orthogonal to a generalized optic axis. The ellipse of $\boldsymbol{E}$ is then similar, and similarly situated, to a critical section of the κ-metric ellipsoid, while the ellipse of $\boldsymbol{H}$ is similar, and similarly situated, to a critical section of the μ-metric ellipsoid. This wave is the only one corresponding to the double root when none of the components of the bivector $\boldsymbol{C}$ in the basis $\boldsymbol{V}^i_*$ is zero.

Finally, in §9.4, the possibility of inhomogeneous waves with **E** and **D**, or **B** and **H** linearly polarized is considered. It is shown that these waves may propagate when one of the components of the bivector $\boldsymbol{C}$ is the basis $\boldsymbol{V}^i_*$ in zero.

9.1. *Secular equation – propagation condition*

Let $\boldsymbol{C}$ be referred to the basis $\boldsymbol{V}^i_*$. We write

$$\boldsymbol{C} = C_i\,\boldsymbol{V}^i_*. \tag{9.1}$$

Then the quadratic forms $\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C}$, $\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C}$, $\boldsymbol{C}^{\mathrm{T}}\kappa\mu^{-1}\kappa\boldsymbol{C}$, $\boldsymbol{C}^{\mathrm{T}}\mu\kappa^{-1}\mu\boldsymbol{C}$ entering the secular equation (3.15) and (3.16) may be written as sums of squares as follows:

$$\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C} = \sum_{i=1}^{3} k_i\,C_i^2, \quad \boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C} = \sum_{i=1}^{3} m_i\,C_i^2, \tag{9.2a, b}$$

$$\boldsymbol{C}^{\mathrm{T}}\kappa\mu^{-1}\kappa\boldsymbol{C} = \sum_{i=1}^{3} m_i^{-1}k_i^2\,C_i^2, \quad \boldsymbol{C}^{\mathrm{T}}\mu\kappa^{-1}\mu\boldsymbol{C} = \sum_{i=1}^{3} k_i^{-1}m_i^2\,C_i^2. \tag{9.3a, b}$$

So, using (8.15), the quadratic form $\boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{C}$ where $\boldsymbol{\Phi}$ is defined by (6.2) or (6.3) may also be written as a sum of squares:

$$\boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{C} = (k_2^{-1}m_3^{-1}+k_3^{-1}m_2^{-1})\,C_1^2+(k_3^{-1}m_1^{-1}+k_1^{-1}m_3^{-1})\,C_2^2+(k_1^{-1}m_2^{-1}+k_2^{-1}m_1^{-1})\,C_3^2. \tag{9.4}$$

Thus the secular equation (3.15), (3.16) reads

$$N^{-4}-N^{-2}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{C}+(\det\kappa)^{-1}\,(\det\mu)^{-1}(\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C})\,(\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C})=0, \tag{9.5}$$

where the quadratic forms $\boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{C}$, $\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C}$, $\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C}$ must be replaced by their expressions (9.4) and (9.2).

When $\boldsymbol{C}$ is such that $\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C}\neq 0$, it may also be written as

$$\sum_{i=1}^{3}\frac{m_i\,k_i\,C_i^2}{N^2 m_i(\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C})-k_i\det\mu}=0, \tag{9.6}$$

which generalizes the Fresnel's equation given in Born & Wolf (1980). Analogously, when $\boldsymbol{C}$ is such that $\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C}\neq 0$, it may also be written as

$$\sum_{i=1}^{3}\frac{k_i\,m_i\,C_i^2}{N^2 k_i(\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C})-m_i\det\kappa}=0. \tag{9.7}$$

Finally, using (9.2), (9.4) and (8.15), it may be checked that (9.5) is equivalent to

$$\{N^{-2}-\lambda_2^{-1}\,(\det\mu)^{-1}(\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C})\}\{N^{-2}-\lambda_2\,(\det\kappa)^{-1}(\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C})\}\\ -N^{-2}m_2\,(\det\kappa)^{-1}(\lambda_1-\lambda_2)\,(\lambda_2-\lambda_3)\,C_2^2=0, \tag{9.8}$$

or

$$\{N^{-2}-\lambda_2^{-1}\,(\det\mu)^{-1}(\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C})\}\{N^{-2}-\lambda_2\,(\det\kappa)^{-1}(\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C})\}\\ -N^{-2}k_2\,(\det\mu)^{-1}(\lambda_2^{-1}-\lambda_1^{-1})\,(\lambda_3^{-1}-\lambda_2^{-1})\,C_2^2=0. \tag{9.9}$$

As, from (9.2) and (9.4), it is clear that the secular equation (9.5) remains unchanged by cyclic permutations of the indices 1, 2, 3, other forms of the secular equation may be written down by cyclic permutations of the indices in (9.8) or (9.9). From (9.8) and (9.9), and the equations obtained from these by cyclic permutations of the indices, it turns out that the secular equations factors when one of the components of the bivector $\boldsymbol{C}$ in the basis $\boldsymbol{V}_*^i$ is zero.

Now, let $\boldsymbol{E}$ and $\boldsymbol{H}$ be also referred to the basis $\boldsymbol{V}_*^i$:

$$\boldsymbol{E}=E_i\,\boldsymbol{V}_*^i,\quad \boldsymbol{H}=H_i\,\boldsymbol{V}_*^i. \tag{9.10}$$

Then, owing to (8.15) and (8.16), the propagation conditions (3.10) and (3.11) for $\boldsymbol{E}$ and $\boldsymbol{H}$, written in components, read

$$\left.\begin{aligned}(N^{-2}k_1\,m_2\,m_3-m_2\,C_2^2-m_3\,C_3^2)\,E_1+C_1C_2m_2\,E_2+C_1\,C_3\,m_3\,E_3&=0,\\ C_2\,C_1\,m_1\,E_1+(N^{-2}k_2\,m_3\,m_1-m_3\,C_3^2-m_1\,C_1^2)\,E_2+C_2\,C_3\,m_3\,E_3&=0,\\ C_3\,C_1\,m_1\,E_1+C_3\,C_2\,m_2\,E_2+(N^{-2}k_3\,m_1\,m_2-m_1\,C_1^2-m_2\,C_2^2)\,E_3&=0,\end{aligned}\right\} \tag{9.11}$$

and, by interchanging the roles of k_1, k_2, k_3 and m_1, m_2, m_3,

$$\left.\begin{aligned}(N^{-2}m_1\,k_2\,k_3-k_2\,C_2^2-k_3\,C_3^2)\,H_1+C_1\,C_2\,k_2\,H_2+C_1\,C_3\,k_3\,H_3&=0,\\ C_2\,C_1\,k_1\,H_1+(N^{-2}m_2\,k_3\,k_1-k_3\,C_3^2-k_1\,C_1^2)\,H_2+C_2\,C_3\,k_3\,H_3&=0,\\ C_3\,C_1\,k_1\,H_1+C_3\,C_2\,k_2\,H_2+(N^{-2}m_3\,k_1\,k_2-k_1\,C_1^2-k_2\,C_2^2)\,H_3&=0.\end{aligned}\right\} \tag{9.12}$$

9.2. *Zero roots*

As pointed out in §3, the secular equation (3.12) or (3.13) for N^{-2} has always one zero root. The two other ones are the roots of the quadratic equation (9.5) for N^{-2}. Here we seek the bivectors $\boldsymbol{C}$ such that this equation has two zero roots (no propagation), or one zero root (only one mode of propagation).

9.2.1. *Both roots zero – no propagation*

There are two cases when equation (9.5) has two zero roots, namely when $\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C} = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{C} = 0$ or when $\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C} = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{C} = 0$.

Case 1: $\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C} = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{C} = 0$. From (9.2*a*) and (9.4) we find that when $\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C} = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{C} = 0$, the bivector $\boldsymbol{C}$ is given, up to a scalar factor, by $\boldsymbol{C}_{\pm}^{(1)}$, defined by

$$\boldsymbol{C}_{\pm}^{(1)} = \pm k_2 k_3 \gamma \boldsymbol{V}_{*}^{1} + \mathrm{i}k_3 k_1 \boldsymbol{V}_{*}^{2} + k_1 k_2 \alpha \boldsymbol{V}_{*}^{3}, \tag{9.13}$$

or their complex conjugates, with γ and α defined by (8.23).

Then using (8.26), we note
$$\boldsymbol{n}_{+}^{\mathrm{T}}\kappa\boldsymbol{C}_{+}^{(1)} = 0, \quad \boldsymbol{n}_{-}^{\mathrm{T}}\kappa\boldsymbol{C}_{-}^{(1)} = 0, \tag{9.14a, b}$$

and from the definition (8.25) of $\boldsymbol{A}_{\pm}$, we find

$$\boldsymbol{A}_{+}\cdot\boldsymbol{C}_{+}^{(1)} = 0, \quad \boldsymbol{A}_{-}\cdot\boldsymbol{C}_{-}^{(1)} = 0. \tag{9.15}$$

We also have
$$\kappa\boldsymbol{C}_{\pm}^{(1)} = (\det\kappa)\,\boldsymbol{A}_{\pm}. \tag{9.16}$$

Using the Appendix B (remark 2), we note that $\boldsymbol{C}_{+}^{(1)\mathrm{T}}\kappa\boldsymbol{C}_{+}^{(1)} = 0$ $(\boldsymbol{C}_{-}^{(1)\mathrm{T}}\kappa\boldsymbol{C}_{-}^{(1)} = 0)$ means that the ellipse of $\boldsymbol{C}_{+}^{(1)}$ $(\boldsymbol{C}_{-}^{(1)})$ is similar, and similarly situated, to an elliptical section of the κ-metric ellipsoid. Also (9.14*a*, *b*) means that the plane of the bivector $\boldsymbol{C}_{+}^{(1)}$ $(\boldsymbol{C}_{-}^{(1)})$ is conjugate to the direction of the real vector $\boldsymbol{n}_{+}$ $(\boldsymbol{n}_{-})$ with respect to the κ-metric ellipsoid. Further, because the generalized optic axis $\boldsymbol{n}_{+}$ $(\boldsymbol{n}_{-})$ is orthogonal to the plane of $\boldsymbol{A}_{+}$ $(\boldsymbol{A}_{-})$, (9.15) means that the projection of the ellipse of $\boldsymbol{C}_{+}^{(1)}$ $(\boldsymbol{C}_{-}^{(1)})$ onto the plane orthogonal to the generalized optic axis $\boldsymbol{n}_{+}$ $(\boldsymbol{n}_{-})$ is similar, and similarly situated, to the ellipse of $\boldsymbol{A}_{+}$ $(\boldsymbol{A}_{-})$ when rotated through a quadrant.

Thus, in a biaxial crystal, no wave propagation is possible with a slowness bivector $\boldsymbol{S} = N\boldsymbol{C}$ whose ellipse is similar, and similarly situated, to one of the sections of the κ-metric ellipsoid by the planes conjugate to the generalized optic axis with respect to this ellipsoid. These sections will be called critical elliptical sections of the κ-metric ellipsoid.

Finally, we note that
$$\boldsymbol{C}_{\pm}^{(1)\mathrm{T}}\mu\boldsymbol{C}_{\pm}^{(1)} = (\det\mu)\,\kappa_2^2\alpha^2\gamma^2(\lambda_1-\lambda_3)^2 \neq 0. \tag{9.17}$$

Case 2: $\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C} = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{C} = 0$. Analogously, interchanging the roles of κ and μ, we find that when $\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C} = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{C} = 0$, the bivector $\boldsymbol{C}$ is given, up to a scalar factor, by $\boldsymbol{C}_{\pm}^{(2)}$ defined by

$$\boldsymbol{C}_{\pm}^{(2)} = \pm m_2 m_3 \gamma \boldsymbol{V}_{*}^{1} + \mathrm{i}m_3 m_1 \boldsymbol{V}_{*}^{2} + m_1 m_2 \alpha \boldsymbol{V}_{*}^{3}, \tag{9.18}$$

or their complex conjugates, with γ and α defined by (8.23).

Then using (8.26), we note
$$\boldsymbol{n}_{+}^{\mathrm{T}}\mu\boldsymbol{C}_{+}^{(2)} = 0, \quad \boldsymbol{n}_{-}^{\mathrm{T}}\mu\boldsymbol{C}_{-}^{(2)} = 0, \tag{9.19}$$

and from the definition (8.25) of $\boldsymbol{A}_{\pm}$, we find

$$\boldsymbol{A}_{+}\cdot\boldsymbol{C}_{+}^{(2)} = 0, \quad \boldsymbol{A}_{-}\cdot\boldsymbol{C}_{-}^{(2)} = 0. \tag{9.20}$$

We also have

$$\mu C_{\pm}^{(2)} = (\det \mu)\, A_{\pm}. \tag{9.21}$$

By using the Appendix B as in case 1, we conclude that in a biaxial crystal no wave propagation is possible with a slowness bivector $S = NC$ whose ellipse is similar, and similarly situated, to one of the sections of the μ-metric ellipsoid by the planes conjugate to the generalized optic axes with respect to this ellipsoid. These sections will be called critical elliptical sections of the μ-metric ellipsoid.

Finally, we note that

$$C_{\pm}^{(2)\mathrm{T}} \kappa C_{\pm}^{(2)} = (\det \kappa)\, m_2^2 \alpha^2 \gamma^2 (\lambda_3^{-1} - \lambda_1^{-1})^2 \neq 0. \tag{9.22}$$

From the analysis of cases 1 and 2, we conclude that there are four critical elliptical sections (two sections of the κ-metric ellipsoid, and two sections of the μ-metric ellipsoid) such that no wave propagation is possible with a slowness bivector whose ellipse is similar, and similarly situated, to one of these sections.

Remark. When the crystal is magnetically (electrically) isotropic, $\mu = \mu 1$ $(\kappa = \kappa 1)$, the μ-metric ellipsoid (κ-metric ellipsoid) is a sphere and there are thus two circular critical sections and two elliptical ones. The bivectors $C_{\pm}^{(2)}$ $(C_{\pm}^{(1)})$ are both isotropic, the optic axes $n_{\pm}$ being normal to their planes.

In the case of magnetically isotropic crystals this is consistent with the result of Hayes (1980) who stated that no wave propagation is possible with an isotropic slowness vector in a plane orthogonal to an optic axis. The existence of two elliptical critical sections (case 1) was not noted by Hayes (1980). However, it is clear from his equations (6.31) and (6.32).

9.2.2. *One root zero – one propagation mode*

The equation (9.5) has one zero root when $C^{\mathrm{T}} \kappa C = 0$ and $C^{\mathrm{T}} \Phi C \neq 0$, or when $C^{\mathrm{T}} \mu C = 0$ and $C^{\mathrm{T}} \Phi C \neq 0$.

Case 1: $C^{\mathrm{T}} \kappa C = 0$, $C^{\mathrm{T}} \Phi C \neq 0$. This case occurs when the ellipse of the bivector C is similarly, and similarly situated, to any elliptical section of the κ-metric ellipsoid other than one of the two critical sections. Then, one root of the secular equation (9.5) is zero and, using (6.2), it is seen that the other one is given by

$$N^{-2} = -(\det \kappa)^{-1} C^{\mathrm{T}} \kappa \mu^{-1} \kappa C = -k_2^{-1} k_3^{-1} (\lambda_1 - \lambda_2)\, C_1^2 + k_1^{-1} k_2^{-1} (\lambda_2 - \lambda_3)\, C_3^2. \tag{9.23}$$

Then, using (3.11), (3.1*d*), (3.2), (5.14) and (5.16) and the identity (A 1) of Appendix A, we find

$$\left.\begin{aligned} D &= -NC \times (\mu^{-1} \kappa C) = -N (\det \mu)^{-1} \mu (\mu C \times \kappa C), \\ B &= \kappa C, \quad E = \kappa^{-1} D, \quad H = \mu^{-1} B, \\ 4W &= C^{\mathrm{T}} \kappa \mu^{-1} \kappa \bar{C} + N \bar{N} \det \kappa^{-1} \{ (C^{\mathrm{T}} \kappa \bar{C})\, (C^{\mathrm{T}} \kappa \mu^{-1} \kappa \mu^{-1} \kappa \bar{C}) - (C^{\mathrm{T}} \kappa \mu^{-1} \kappa \bar{C})^2 \}, \\ 2R &= \det \kappa^{-1} \{ (C^{\mathrm{T}} \kappa \mu^{-1} \kappa \mu^{-1} \kappa \bar{C})\, \kappa S^{+} - (C^{\mathrm{T}} \kappa \mu^{-1} \kappa \bar{C})\, \kappa \mu^{-1} \kappa S^{+} \}, \end{aligned}\right\} \tag{9.24}$$

where N is given by (9.23).

Now, $S^{\mathrm{T}} \kappa S = 0$ and $S^{\mathrm{T}} \kappa \mu^{-1} \kappa S = -\det \kappa$. Hence, taking the imaginary parts we have $S^{-\mathrm{T}} \kappa S^{+} = 0$ and $S^{-\mathrm{T}} \kappa \mu^{-1} \kappa S^{+} = 0$, so that S^{+} and S^{-} are conjugate directions with respect to the κ-metric ellipsoid and also with respect to the $\kappa\mu^{-1}\kappa$-metric ellipsoid. It follows immediately that $\boldsymbol{R}\cdot\boldsymbol{S}^- = 0$. We also check that

$$8\boldsymbol{R}\cdot\boldsymbol{S}^+ = -\det\kappa^{-1}(\boldsymbol{C}^{\mathrm{T}}\kappa\mu^{-1}\kappa\bar{\boldsymbol{C}})\,(N\kappa\mu^{-1}\kappa\boldsymbol{C}+\bar{N}\kappa\mu^{-1}\kappa\bar{\boldsymbol{C}})\cdot(N\boldsymbol{C}+\overline{N\boldsymbol{C}})$$
$$+\det\kappa^{-1}(\boldsymbol{C}^{\mathrm{T}}\kappa\mu^{-1}\kappa\mu^{-1}\kappa\bar{\boldsymbol{C}})\,(N\kappa\boldsymbol{C}+\bar{N}\kappa\bar{\boldsymbol{C}})\cdot(N\boldsymbol{C}+\overline{N\boldsymbol{C}}) = 8W.$$

To illustrate these results we present a specific example.

Example. Let
$$\boldsymbol{C} = k_1^{-\frac{1}{2}}\boldsymbol{V}_*^1+\mathrm{i}2^{\frac{1}{2}}k_2^{-\frac{1}{2}}\boldsymbol{V}_*^2+k_3^{-\frac{1}{2}}\boldsymbol{V}_*^3,$$

so that the ellipse of $\boldsymbol{C}$ is similar, and similarly situated, to the section of the κ-metric ellipsoid by the plane orthogonal to the vector $k_1^{\frac{1}{2}}\boldsymbol{V}_1-k_3^{\frac{1}{2}}\boldsymbol{V}_3$.

Then $\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C} = 0$, and $\boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{C} \neq 0$ provided that $2\lambda_2 \neq \lambda_1+\lambda_3$. We here assume $2\lambda_2 > \lambda_1+\lambda_3$. Then, from (9.23),
$$N = (\det\kappa)^{\frac{1}{2}}(2\lambda_2-\lambda_1-\lambda_3)^{-1},$$
and from (9.24),
$$\boldsymbol{D} = (2\lambda_2-\lambda_1-\lambda_3)^{-1}\{\mathrm{i}(\tfrac{1}{2}k_1)^{\frac{1}{2}}(\lambda_2-\lambda_3)\,\boldsymbol{V}_1-k_2^{\frac{1}{2}}(\lambda_1-\lambda_3)\,\boldsymbol{V}_2+\mathrm{i}(\tfrac{1}{2}k_3)^{\frac{1}{2}}(\lambda_1-\lambda_2)\,\boldsymbol{V}_3\},$$
$$\boldsymbol{E} = (2\lambda_2-\lambda_1-\lambda_3)^{-1}\{\mathrm{i}(2k_1)^{-\frac{1}{2}}(\lambda_2-\lambda_3)\,\boldsymbol{V}_*^1-k_2^{-\frac{1}{2}}(\lambda_1-\lambda_3)\,\boldsymbol{V}_*^2+\mathrm{i}(2k_3)^{-\frac{1}{2}}(\lambda_1-\lambda_2)\,\boldsymbol{V}_*^3\},$$
$$\boldsymbol{B} = k_1^{\frac{1}{2}}\boldsymbol{V}_1+\mathrm{i}(\tfrac{1}{2}k_2)^{\frac{1}{2}}\boldsymbol{V}_2+k_3^{\frac{1}{2}}\boldsymbol{V}_3,$$
$$\boldsymbol{H} = m_1^{-1}k_1^{\frac{1}{2}}\boldsymbol{V}_*^1+\mathrm{i}m_2^{-1}(\tfrac{1}{2}k_2)^{\frac{1}{2}}\boldsymbol{V}_*^2+m_3^{-1}k_3^{\frac{1}{2}}\boldsymbol{V}_*^3,$$
$$2W = 2\lambda_1+\lambda_2+2\lambda_3+(2\lambda_2-\lambda_1-\lambda_3)^{-1}\{(\lambda_2-\lambda_3)^2+2(\lambda_1-\lambda_3)^2+(\lambda_1-\lambda_2)^2\},$$
$$2\boldsymbol{R} = (\det\kappa)^{-\frac{1}{2}}(2\lambda_2-\lambda_1-\lambda_3)^{-1}\{k_1^{\frac{1}{2}}(2\lambda_3^2+\lambda_2^2-2\lambda_1\lambda_3-\lambda_1\lambda_2)\,\boldsymbol{V}_1+k_3^{\frac{1}{2}}(2\lambda_2^2+\lambda_1^2-2\lambda_2\lambda_3-\lambda_1\lambda_3)\,\boldsymbol{V}_3\}.$$

Case 2: $\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C} = 0,\ \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{C} \neq 0$

This case occurs when the ellipse of the bivector $\boldsymbol{C}$ is similar, and similarly situated, to any elliptical section of the μ-metric ellipsoid other than one of the two critical sections. Then, one root of the secular equation (9.5) is zero, and, using (6.3), it is seen that the other one is given by
$$N^{-2} = -(\det\mu)^{-1}\boldsymbol{C}^{\mathrm{T}}\mu\kappa^{-1}\mu\boldsymbol{C} = m_2^{-1}m_3^{-1}(\lambda_2^{-1}-\lambda_1^{-1})\,C_1^2-m_1^{-1}m_2^{-1}(\lambda_3^{-1}-\lambda_2^{-1})\,C_3^2. \qquad (9.25)$$

Then, using (3.10), (3.1*c*), (3.2), (5.13), (5.15) and the identity (A 1) of Appendix A, we find

$$\left.\begin{aligned}
\boldsymbol{B} &= N\boldsymbol{C}\times(\kappa\mu^{-1}\boldsymbol{C}) = N(\det\kappa)^{-1}\kappa(\kappa\boldsymbol{C}\times\mu\boldsymbol{C}),\\
\boldsymbol{D} &= \mu\boldsymbol{C},\quad \boldsymbol{H} = \mu^{-1}B,\quad \boldsymbol{E} = \kappa^{-1}\boldsymbol{D},\\
4W &= \boldsymbol{C}^{\mathrm{T}}\mu\kappa^{-1}\mu\bar{\boldsymbol{C}}+N\bar{N}\det\mu^{-1}\{(\boldsymbol{C}^{\mathrm{T}}\mu\bar{\boldsymbol{C}})\,(\boldsymbol{C}^{\mathrm{T}}\mu\kappa^{-1}\mu\kappa^{-1}\mu\bar{\boldsymbol{C}})-(\boldsymbol{C}^{\mathrm{T}}\mu\kappa^{-1}\mu\bar{\boldsymbol{C}})^2\},\\
2\boldsymbol{R} &= \det\mu^{-1}\{(\boldsymbol{C}^{\mathrm{T}}\mu\kappa^{-1}\mu\kappa^{-1}\mu\bar{\boldsymbol{C}})\,\mu\boldsymbol{S}^+-(\boldsymbol{C}^{\mathrm{T}}\mu\kappa^{-1}\mu\bar{\boldsymbol{C}})\,\mu\kappa^{-1}\mu\boldsymbol{S}^+\},
\end{aligned}\right\} \qquad (9.26)$$

where N is given by (9.25). As in case 1 (results (9.23) and (9.24)) we confirm (5.6).

Remark. When the crystal is magnetically (electrically) isotropic, $\mu = \mu 1$ ($\kappa = \kappa 1$), then in case 2 (case 1), $\boldsymbol{C}$ is isotropic, but not in a plane orthogonal to an optic axis. From equations (9.26) ((9.24)), we note that the corresponding single propagation mode is such that $\boldsymbol{D}$ ($\boldsymbol{B}$) and $\boldsymbol{S}$ are parallel and isotropic. The **D** (**B**)-field is thus circularly polarised (see Hayes 1987, §6.5 for the case of magnetically isotropic crystals).

In case 1 (case 2), however, $\boldsymbol{C}$ is not isotropic. The fact that only one wave may propagate in magnetically isotropic crystals with a slowness bivector whose ellipse is similar, and similarly situated, to any section of the κ-metric ellipsoid other than one of the two critical sections was not noted by Hayes (1987). However, it is clear from his equation (6.5).

9.3. *Double roots*: $\boldsymbol{D}$ *and* $\boldsymbol{B}$ *parallel*

The condition that the secular equation (9.5) have a double root is

$$(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{C})^2 = 4\det\kappa^{-1}\det\mu^{-1}(\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C})\,(\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C}). \tag{9.27}$$

Then, as shown in §4.3, two cases may occur. In case 1, corresponding to the double root there is one wave solution obeying (4.21) and (4.22), with $\boldsymbol{D}$ and $\boldsymbol{B}$ parallel and isotropic together with respect to κ^{-1} and μ^{-1}. In case 2, corresponding to the double root there are two different wave solutions and any linear combination of the fields of these two waves is also a wave solution with the same slowness. Here it is shown that, for a biaxial crystal, case 1 occurs when the bivector $\boldsymbol{C}$ satisfying (9.27) has no zero components in the basis $\boldsymbol{V}^i_*$ (general case), whereas case 2 occurs when it has one zero component (special cases). The corresponding wave solutions are obtained.

By using (9.2), (9.4) and (8.15), the condition (9.27) for a double root may be written in the form

$$\{k_1(\lambda_2-\lambda_3)\,C_1^2+k_2(\lambda_1-\lambda_3)\,C_2^2+k_3(\lambda_1-\lambda_2)\,C_3^2\}^2 = 4k_1k_3(\lambda_1-\lambda_2)\,(\lambda_2-\lambda_3)\,C_1^2C_3^2, \tag{9.28}$$

or, equivalently

$$\{m_1(\lambda_3^{-1}-\lambda_2^{-1})\,C_1^2+m_2(\lambda_3^{-1}-\lambda_1^{-1})\,C_2^2+m_3(\lambda_2^{-1}-\lambda_1^{-1})\,C_3^2\}^2 = 4m_1m_3(\lambda_2^{-1}-\lambda_1^{-1})\,(\lambda_3^{-1}-\lambda_2^{-1})\,C_1^2C_3^2. \tag{9.29}$$

Equations (9.28) and (9.29) may also be written

$$\{\gamma^2C_1^2+C_2^2+\alpha^2C_3^2\}^2 = 4\alpha^2\gamma^2C_1^2C_3^2, \tag{9.30}$$

with γ and α defined by (8.23). There are thus two pairs of possibilities:

$$\gamma C_1+\alpha C_3 = \mp\mathrm{i}C_2, \tag{9.31}$$

that is

$$\boldsymbol{A}_+\cdot\boldsymbol{C} = 0 \quad\text{or}\quad \bar{\boldsymbol{A}}_+\cdot\boldsymbol{C} = 0, \tag{9.32}$$

and

$$\gamma C_1-\alpha C_3 = \pm\mathrm{i}C_2, \tag{9.33}$$

that is

$$\boldsymbol{A}_-\cdot\boldsymbol{C} = 0 \quad\text{or}\quad \bar{\boldsymbol{A}}_-\cdot\boldsymbol{C} = 0. \tag{9.34}$$

We assume here that C_2 is not equal to zero for otherwise from equation (9.31) or (9.33) the ratio C_1/C_3 is real so that $\boldsymbol{C}$ is a linear bivector and hence the corresponding wave is homogeneous, a case which has been studied in §8. Then, for cases (9.31) and (9.33) the double eigenvalue $(-N^{-2})$ of the eigenvalue problems (3.4) and (3.6) is given respectively by

$$N^{-2} = (k_3^{-1}\alpha C_1-k_1^{-1}\gamma C_3)\,(m_3^{-1}\alpha C_1-m_1^{-1}\gamma C_3), \tag{9.35}$$

or

$$N^{-2} = (k_3^{-1}\alpha C_1+k_1^{-1}\gamma C_3)\,(m_3^{-1}\alpha C_1+m_1^{-1}\gamma C_3), \tag{9.36}$$

and the system (9.11) for the determination of the corresponding eigenbivectors $\boldsymbol{E}$ reduces respectively to

$$\left.\begin{aligned}
&C_1\{\lambda_3k_1C_1-\alpha\gamma k_2(\lambda_1-\lambda_3)\,C_3\}\,E_1+\lambda_3C_1C_2k_2E_2+\lambda_2C_1C_3k_3E_3 = 0,\\
&\lambda_3C_1k_1E_1\pm\mathrm{i}k_2(\gamma\lambda_3C_1+\alpha\lambda_1C_3)\,E_2+\lambda_1C_3k_3E_3 = 0,\\
&\lambda_2C_3C_1k_1E_1+\lambda_1C_3C_2k_2E_2+C_3\{\lambda_1k_3C_3+\alpha\gamma k_2(\lambda_1-\lambda_3)\,C_1\}\,E_3 = 0,
\end{aligned}\right\} \tag{9.37 a–c}$$

or

$$\left.\begin{aligned} & C_1\{\lambda_3 k_1 C_1+\alpha\gamma k_2(\lambda_1-\lambda_3)\,C_3\}\,E_1+\lambda_3 C_1 C_2 k_2 E_2+\lambda_2 C_1 C_3 k_3 E_3=0,\\ & \lambda_3 C_1 k_1 E_1 \pm \mathrm{i}k_2(-\gamma\lambda_3 C_1+\alpha\lambda_1 C_3)\,E_2+\lambda_1 C_3 k_3 E_3=0,\\ & \lambda_2 C_3 C_1 k_1 E_1+\lambda_1 C_3 C_2 k_2 E_2+C_3\{\lambda_1 k_3 C_3-\alpha\gamma k_2(\lambda_1-\lambda_3)\,C_1\}\,E_3=0.\end{aligned}\right\} \tag{9.38a–c}$$

The corresponding forms of the system (9.12) for the determination of the eigenbivectors $\boldsymbol{H}$ may be read off from (9.37) and (9.38) by interchanging the roles of k_1, k_2, k_3 and m_1, m_2, m_3.

9.3.1. *General case*: $C_1 \neq 0$, $C_2 \neq 0$, $C_3 \neq 0$

For case (9.31), when $C_1 \neq 0$, $C_2 \neq 0$, $C_3 \neq 0$, we note using the constitutive equation $D^i = k_i E_i$ (no sum) and $\boldsymbol{C}\cdot\boldsymbol{D} = 0$, that equation (9.37 a) gives $\alpha D^1 - \gamma D^3 = 0$, or equivalently,

$$\boldsymbol{n}_+\cdot\boldsymbol{D} = 0. \tag{9.39}$$

Then, using $\boldsymbol{C}\cdot\boldsymbol{D} = 0$, it may be shown that if the upper $(-)$ sign is taken in (9.31), then $\boldsymbol{D}$ is parallel to $\boldsymbol{A}_+$ defined by (8.25), whilst if the lower $(+)$ sign is taken, then $\boldsymbol{D}$ is parallel to $\bar{\boldsymbol{A}}_+$. Thus there is a simple infinity of eigenbivectors $\boldsymbol{E}$ (one wave solution) corresponding to the double eigenvalue $(-N^{-2})$ given by (9.35).

On choosing the upper $(-)$ sign in equation (9.31) we find

$$\left.\begin{aligned} & \boldsymbol{D} = (m_3^{-1}\alpha C_1 - m_1^{-1}\gamma C_3)^{\frac{1}{2}}\boldsymbol{A}_+, \quad \boldsymbol{B} = \mathrm{i}(k_3^{-1}\alpha C_1 - k_1^{-1}\gamma C_3)^{\frac{1}{2}}\boldsymbol{A}_+,\\ & \boldsymbol{E} = (\det\kappa)^{-1}(m_3^{-1}\alpha C_1 - m_1^{-1}\gamma C_3)^{\frac{1}{2}}\,\boldsymbol{C}_+^{(1)},\\ & \boldsymbol{H} = \mathrm{i}(\det\mu)^{-1}(k_3^{-1}\alpha C_1 - k_1^{-1}\gamma C_3)^{\frac{1}{2}}\,\boldsymbol{C}_+^{(2)},\\ & 2W = k_2^{-1}|m_3^{-1}\alpha C_1 - m_1^{-1}\gamma C_3| + m_2^{-1}|k_3^{-1}\alpha C_1 - k_1^{-1}\gamma C_3|,\\ & 2\boldsymbol{R} = \det\kappa^{-1}\det\mu^{-1}\{(m_3^{-1}\alpha C_1 - m_1^{-1}\gamma C_3)^{\frac{1}{2}}(k_3^{-1}\alpha\bar{C}_1 - k_1^{-1}\gamma\bar{C}_3)^{\frac{1}{2}}\,\boldsymbol{C}_+^{(1)}\times\bar{\boldsymbol{C}}_+^{(2)}\}^-,\end{aligned}\right\} \tag{9.40}$$

where $\boldsymbol{A}_+$, $\boldsymbol{C}_+^{(1)}$, $\boldsymbol{C}_+^{(2)}$ are defined by (8.25), (9.13) and (9.18). In the derivation use has been made of (9.16) and (9.21) and the identity (8.24). Thus both the ellipses of $\boldsymbol{D}$ and $\boldsymbol{B}$ are similar and similarly situated to either of the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane Π_+ orthogonal to the generalized optic axis $\boldsymbol{n}_+$. The ellipses of $\boldsymbol{E}$ and $\boldsymbol{H}$ are similar, and similarly situated, to the critical sections respectively of the κ-metric ellipsoid and of the μ-metric ellipsoid by the planes conjugate to $\boldsymbol{n}_+$ with respect to these ellipsoids.

Analogously, on choosing the lower $(+)$ sign in equation (9.31) we find

$$\left.\begin{aligned} & \boldsymbol{D} = (m_3^{-1}\alpha C_1 - m_1^{-1}\gamma C_3)^{\frac{1}{2}}\bar{\boldsymbol{A}}_+, \quad \boldsymbol{B} = -\mathrm{i}(k_3^{-1}\alpha C_1 - k_1^{-1}\gamma C_3)^{\frac{1}{2}}\bar{\boldsymbol{A}}_+,\\ & \boldsymbol{E} = (\det\kappa)^{-1}(m_3^{-1}\alpha C_1 - m_1^{-1}\gamma C_3)^{\frac{1}{2}}\bar{\boldsymbol{C}}_+^{(1)},\\ & \boldsymbol{H} = -\mathrm{i}(\det\mu)^{-1}(k_3^{-1}\alpha C_1 - k_1^{-1}\gamma C_3)\,\bar{\boldsymbol{C}}_+^{(2)},\\ & 2W = k_2^{-1}|m_3^{-1}\alpha C_1 - m_1^{-1}\gamma C_3| + m_2^{-1}|k_3^{-1}\alpha C_1 - k_1^{-1}\gamma C_3|,\\ & 2\boldsymbol{R} = \det\kappa^{-1}\det\mu^{-1}\{(m_3^{-1}\alpha C_1 - m_1^{-1}\gamma C_3)^{\frac{1}{2}}(k_3^{-1}\alpha\bar{C}_1 - k_1^{-1}\gamma\bar{C}_3)^{\frac{1}{2}}\boldsymbol{C}_+^{(1)}\times\bar{\boldsymbol{C}}_+^{(2)}\}^-.\end{aligned}\right\} \tag{9.41}$$

The geometrical interpretation is the same as for (9.40) except that the ellipses of $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{H}$ have now the opposite handedness.

For case (9.33), when $C_1 \neq 0$, $C_2 \neq 0$, $C_3 \neq 0$, we now find from (9.38 a) that $\alpha D^1 + \gamma D^3 = 0$, or equivalently

$$\boldsymbol{n}_-\cdot\boldsymbol{D} = 0. \tag{9.42}$$

As in case (9.31) it may be shown that if the upper $(+)$ sign is taken in (9.33), then $\boldsymbol{D}$ is parallel to $\boldsymbol{A}_-$ defined by (8.25), whereas if the lower $(-)$ sign is taken, then $\boldsymbol{D}$ is parallel to $\bar{\boldsymbol{A}}_-$.

On choosing the upper $(+)$ sign in equation (9.33) we find

$$\left.\begin{aligned} \boldsymbol{D} &= (m_3^{-1}\alpha C_1 + m_1^{-1}\gamma C_3)^{\frac{1}{2}}\boldsymbol{A}_-, \quad \boldsymbol{B} = \mathrm{i}(k_3^{-1}\alpha C_1 + k_1^{-1}\gamma C_3)^{\frac{1}{2}}\boldsymbol{A}_-, \\ \boldsymbol{E} &= (\det \kappa)^{-1}(m_3^{-1}\alpha C_1 + m_1^{-1}\gamma C_3)^{\frac{1}{2}}\boldsymbol{C}_-^{(1)}, \\ \boldsymbol{H} &= \mathrm{i}(\det \mu)^{-1}(k_3^{-1}\alpha C_1 + k_1^{-1}\gamma C_3)^{\frac{1}{2}}\boldsymbol{C}_-^{(2)}, \\ 2W &= k_2^{-1}|m_3^{-1}\alpha C_1 + m_1^{-1}\gamma C_3| + m_2^{-1}|k_3^{-1}\alpha C_1 + k_1^{-1}\gamma C_3|, \\ 2\boldsymbol{R} &= \det \kappa^{-1}\det \mu^{-1}\{(m_3^{-1}\alpha C_1 + m_1^{-1}\gamma C_3)^{\frac{1}{2}}(k_3^{-1}\alpha \bar{C}_1 + k_1^{-1}\gamma \bar{C}_3)^{\frac{1}{2}}\boldsymbol{C}_-^{(1)} \times \bar{\boldsymbol{C}}_-^{(2)}\}^-. \end{aligned}\right\} \tag{9.43}$$

where $\boldsymbol{A}_-$, $\boldsymbol{C}_-^{(1)}$, $\boldsymbol{C}_-^{(2)}$ are defined by (8.25), (9.13) and (9.18). In the derivation use has been made of (9.16), (9.21) and the identity (8.24). Thus both the ellipses of $\boldsymbol{D}$ and $\boldsymbol{B}$ are similar, and similarly situated, to either of the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane Π_- orthogonal to the generalized optic axis $\boldsymbol{n}_-$. The ellipses of $\boldsymbol{E}$ and $\boldsymbol{H}$ are similar, and similarly situated, to the critical sections respectively of the κ-metric ellipsoid and of the μ-metric ellipsoid by the planes conjugate to $\boldsymbol{n}_-$ with respect to these ellipsoids.

Analogously on choosing the lower $(-)$ sign in equation (9.33) we find

$$\left.\begin{aligned} \boldsymbol{D} &= (m_3^{-1}\alpha C_1 + m_1^{-1}\gamma C_3)^{\frac{1}{2}}\bar{\boldsymbol{A}}_-, \quad \boldsymbol{B} = -\mathrm{i}(k_3^{-1}\alpha C_1 + k_1^{-1}\gamma C_3)^{\frac{1}{2}}\bar{\boldsymbol{A}}_-, \\ \boldsymbol{E} &= (\det \kappa)^{-1}(m_3^{-1}\alpha C_1 + m_1^{-1}\gamma C_3)^{\frac{1}{2}}\bar{\boldsymbol{C}}_-^{(1)}, \\ \boldsymbol{H} &= -\mathrm{i}(\det \mu)^{-1}(k_3^{-1}\alpha C_1 + k_1^{-1}\gamma C_3)^{\frac{1}{2}}\bar{\boldsymbol{C}}_-^{(2)}, \\ 2W &= k_2^{-1}|m_3^{-1}\alpha C_1 + m_1^{-1}\gamma C_3| + m_2^{-1}|k_3^{-1}\alpha C_1 + k_1^{-1}\gamma C_3|, \\ 2\boldsymbol{R} &= \det \kappa^{-1}\det \mu^{-1}\{(m_3^{-1}\alpha C_1 + m_1^{-1}\gamma C_3)^{\frac{1}{2}}(k_3^{-1}\alpha \bar{C}_1 + k_1^{-1}\gamma \bar{C}_3)^{\frac{1}{2}}\boldsymbol{C}_-^{(1)} \times \bar{\boldsymbol{C}}_-^{(2)}\}^-. \end{aligned}\right\} \tag{9.44}$$

The geometrical interpretation is the same as for (9.43) except that the ellipses of $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{H}$ have now the opposite handedness.

Remark. For a magnetically (electrically) isotropic crystal, $\mu = \mu 1$ ($\kappa = \kappa 1$), $\boldsymbol{A}_+$ and $\boldsymbol{A}_-$ are isotropic (see equation (8.25)), and the optic axes $\boldsymbol{n}_+$ and $\boldsymbol{n}_-$ are orthogonal to their planes. Also from equation (9.21) and (9.16), the bivectors $\boldsymbol{C}_\pm^{(2)}$ ($\boldsymbol{C}_\pm^{(1)}$) are parallel to the bivectors $\boldsymbol{A}_\pm$. It follows that for each of the solutions (9.40), (9.41), (9.43) and (9.44), the amplitude bivectors $\boldsymbol{D}$, $\boldsymbol{B}$ and $\boldsymbol{H}$ ($\boldsymbol{D}$, $\boldsymbol{B}$ and $\boldsymbol{E}$) are parallel and isotropic. Such solutions have been considered by Hayes (1987, §6.4) for the case of magnetically isotropic crystals.

9.3.2. *Special cases*

In this section we consider the possibility of one of the components C_1, C_2, C_3 being zero. We note first that if $C_2 = 0$, it follows from either (9.31) or (9.33) that $\boldsymbol{C}$ is a linear bivector either along the generalized optic axis $\boldsymbol{n}_+$ or along the generalize optic axis $\boldsymbol{n}_-$. Propagation of homogeneous waves along a generalized optic axis has been studied in §8: corresponding to the double eigenvalue $(-N^{-2})$ of the eigenvalue problem (3.4) or (3.6) there is a double infinity of eigenbivectors $\boldsymbol{E}$ or $\boldsymbol{H}$ (see analytical results of §8.4). Accordingly we now deal with the two possibilities:

$$C_3 = 0, \quad \gamma C_1 = \mp \mathrm{i} C_2, \tag{9.45}$$

$$C_1 = 0, \quad \alpha C_3 = \mp \mathrm{i} C_2. \tag{9.46}$$

For both possibilities (9.31) and (9.33) coalesce. Then one of the equations (9.37) is satisfied identically and the other two reduce to $\boldsymbol{C}\cdot\boldsymbol{D} = 0$. There is thus a double infinity of eigenbivectors $\boldsymbol{D}$ corresponding to the double eigenvalue $(-N^{-2})$ given by (9.35) or (9.36). Then, as shown in §4.3 (case 2), corresponding to N^{-2} there are two wave solutions $\boldsymbol{D}_1$, $\boldsymbol{E}_1$, $\boldsymbol{B}_1$, $\boldsymbol{H}_1$ and $\boldsymbol{D}_2$, $\boldsymbol{E}_2$, $\boldsymbol{B}_2$, $\boldsymbol{H}_2$ satisfying the orthogonality relations of §4.2, and with the fields $\mathbf{D}_1$, $\mathbf{E}_1$ and $\mathbf{B}_2$, $\mathbf{H}_2$ linearly polarized. Any linear combination of the amplitudes of these two waves defines a wave with the same slowness $\boldsymbol{S} = N\boldsymbol{C}$.

Possibility (9.45): $C_3 = 0,\ \gamma C_1 = \mp \mathrm{i} C_2$

For the two wave solutions we find

$$\left.\begin{aligned} \boldsymbol{D}_1 &= k_3^{\frac{1}{2}}\alpha \boldsymbol{V}_3, \quad \boldsymbol{B}_1 = \pm \mathrm{i} m_3^{\frac{1}{2}}(\gamma \boldsymbol{V}_1 \pm \mathrm{i}\boldsymbol{V}_2),\\ \boldsymbol{E}_1 &= k_3^{-\frac{1}{2}}\alpha \boldsymbol{V}_*^3, \quad \boldsymbol{H}_1 = \pm \mathrm{i} m_3^{\frac{1}{2}}(\gamma m_1^{-1}\boldsymbol{V}_*^1 \pm \mathrm{i} m_2^{-1}\boldsymbol{V}_*^2),\\ W_1 &= (2m_2)^{-1}m_3, \quad \boldsymbol{R}_1 = (2m_2)^{-1}\lambda_3^{-\frac{1}{2}}\alpha \boldsymbol{V}_1, \end{aligned}\right\} \tag{9.47}$$

and

$$\left.\begin{aligned} \boldsymbol{D}_2 &= \mp \mathrm{i} k_3^{\frac{1}{2}}(\gamma \boldsymbol{V}_1 \pm \mathrm{i}\boldsymbol{V}_2), \quad \boldsymbol{B}_2 = m_3^{\frac{1}{2}}\alpha \boldsymbol{V}_3,\\ \boldsymbol{E}_2 &= \mp \mathrm{i} k_3^{\frac{1}{2}}(\gamma k_1^{-1}\boldsymbol{V}_*^1 \pm \mathrm{i} k_2^{-1}\boldsymbol{V}_*^2), \quad \boldsymbol{H}_2 = m_3^{-\frac{1}{2}}\alpha \boldsymbol{V}_*^3,\\ W_2 &= (2k_2)^{-1}k_3, \quad \boldsymbol{R}_2 = (2k_2)^{-1}\lambda_3^{\frac{1}{2}}\alpha \boldsymbol{V}_1. \end{aligned}\right\} \tag{9.48}$$

It is easy to check that the relations (4.15) to (4.20) are satisfied. Both waves have the common slowness $\boldsymbol{S}$ given by

$$\boldsymbol{S} = \alpha^{-1}(k_3 m_3)^{\frac{1}{2}}(\boldsymbol{V}_*^1 \pm \mathrm{i}\gamma \boldsymbol{V}_*^2), \tag{9.49}$$

and both waves have their mean energy flux vector along $\boldsymbol{V}_1$.

For a combined wave solution $\boldsymbol{D} = a\boldsymbol{D}_1 + b\boldsymbol{D}_2$, $\boldsymbol{E} = a\boldsymbol{E}_1 + b\boldsymbol{E}_2$, etc., we find that the weighted energy density is given by

$$W = (2k_1 m_1)^{-1}(|a|^2 k_1 m_3 + |b|^2 k_3 m_1), \tag{9.50}$$

in accordance with (5.22), and that the weighted energy flux is given by

$$\boldsymbol{R} = |a|^2\boldsymbol{R}_1 + |b|^2\boldsymbol{R}_2 \pm \tfrac{1}{2}(a\bar{b})^{-}\gamma(k_3 m_3)^{\frac{1}{2}}(k_1^{-1}m_2^{-1} + k_2^{-1}m_1^{-1})\,\boldsymbol{V}_3. \tag{9.51}$$

The interaction term in $\boldsymbol{R}$ is orthogonal to $\boldsymbol{V}_*^1$ and $\boldsymbol{V}_*^2$ in accordance with (5.26).

Remark 1. *Energy flux for combined waves*

The weighted energy flux $\boldsymbol{R}$ given by (9.51) may be written

$$\boldsymbol{R} = \tfrac{1}{2}(k_3 m_3)^{\frac{1}{2}}(\hat{a}^2 + \hat{b}^2)\,\{\alpha \boldsymbol{V}_1 \pm \sin 2\varphi \sin(\theta - \delta)\, h\boldsymbol{V}_3\}, \tag{9.52}$$

where

$$a = |a|\,\mathrm{e}^{\mathrm{i}\theta}, \quad b = |b|\,\mathrm{e}^{\mathrm{i}\delta}, \quad \hat{a} = |a|/(k_3 m_2)^{\frac{1}{2}}, \quad \hat{b} = |b|/(k_2 m_3)^{\frac{1}{2}}, \tag{9.53}$$
$$\mathrm{tg}\,\varphi = \hat{a}/\hat{b}, \quad 2h = \gamma(k_2 k_3 m_2 m_3)^{\frac{1}{2}}(k_1^{-1}m_2^{-1} + k_2^{-1}m_1^{-1}).$$

As $|a|$, $|b|$, θ and δ are varied, $\boldsymbol{R}$ varies from lying along $\alpha \boldsymbol{V}_1 + h\boldsymbol{V}_3$ to lying along $\alpha\boldsymbol{V}_1 - h\boldsymbol{V}_3$ and may take any intermediate direction between these two extremes.

Remark 2. *Waves with* $\boldsymbol{D}$ *and* $\boldsymbol{B}$ *parallel*

We note that for $b = \mathrm{i}a$, the combined wave reduces to the wave (9.40) with $C_3 = 0$ when the upper signs are chosen in (9.47), (9.48) and to the wave (9.43) when the lower signs are chosen. Also for $b = -\mathrm{i}a$, the combined wave reduces to the wave (9.44) with $C_3 = 0$ when the upper signs are chosen in (9.47), (9.48) and to the wave (9.41) with $C_3 = 0$ when the lower sign is chosen.

Thus, when the upper signs are chosen in (9.47) and (9.48) the amplitude bivectors of any combined wave may also be written as linear combinations of the amplitudes (9.40) and (9.44). When the lower signs are chosen the amplitude bivectors of any combined wave may also be written as linear combinations of the amplitudes (9.43) and (9.41). The amplitudes of any combined wave are thus also linear combinations of the amplitudes of two waves for each of which $\boldsymbol{D}$ and $\boldsymbol{B}$ are parallel. For one wave the ellipses of $\boldsymbol{D}$ and $\boldsymbol{B}$ are similar, and similarly situated, to either of the elliptical section of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane Π_+ orthogonal to the generalized optic axis $\boldsymbol{n}_+$, and for the other the ellipses of $\boldsymbol{D}$ and $\boldsymbol{B}$ are similar, and similarly situated, to either of the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane Π_- orthogonal to the generalized optic axis $\boldsymbol{n}_-$.

Possibility (9.46): $C_1 = 0$, $\alpha C_3 = \mp \mathrm{i} C_2$

Analogously, for the two wave solutions, we find

$$\left.\begin{aligned} &\boldsymbol{D}_3 = k_1^{\frac{1}{2}} \gamma \boldsymbol{V}_1, \quad \boldsymbol{B}_3 = \mp \mathrm{i} m_1^{\frac{1}{2}} (\pm \mathrm{i} \boldsymbol{V}_2 + \alpha \boldsymbol{V}_3), \\ &\boldsymbol{E}_3 = k_1^{-\frac{1}{2}} \gamma \boldsymbol{V}_*^1, \quad \boldsymbol{H}_3 = \mp \mathrm{i} m_1^{\frac{1}{2}} (\pm \mathrm{i} m_2^{-1} \boldsymbol{V}_*^2 + \alpha m_3^{-1} \boldsymbol{V}_*^3), \\ &W_3 = (2m_2)^{-1} m_1, \quad \boldsymbol{R}_3 = (2m_2)^{-1} \lambda_1^{-\frac{1}{2}} \gamma \boldsymbol{V}_3, \end{aligned}\right\} \tag{9.54}$$

and

$$\left.\begin{aligned} &\boldsymbol{D}_4 = \pm \mathrm{i} k_1^{\frac{1}{2}} (\pm \mathrm{i} \boldsymbol{V}_2 + \alpha \boldsymbol{V}_3), \quad \boldsymbol{B}_4 = m_1^{\frac{1}{2}} \gamma \boldsymbol{V}_1, \\ &\boldsymbol{E}_4 = \pm \mathrm{i} k_1^{\frac{1}{2}} (\pm \mathrm{i} k_2^{-1} \boldsymbol{V}_*^2 + \alpha k_3^{-1} \boldsymbol{V}_*^3), \quad \boldsymbol{H}_4 = m_1^{-\frac{1}{2}} \gamma \boldsymbol{V}_*^1, \\ &W_3 = (2k_2)^{-1} k_1, \quad \boldsymbol{R}_4 = (2k_2)^{-1} \lambda_1^{\frac{1}{2}} \gamma \boldsymbol{V}_3. \end{aligned}\right\} \tag{9.55}$$

Both waves have the common slowness $\boldsymbol{S}$ given by

$$\boldsymbol{S} = \gamma^{-1} (k_1 m_1)^{\frac{1}{2}} (\pm \mathrm{i} \alpha \boldsymbol{V}_*^2 + \boldsymbol{V}_*^3), \tag{9.56}$$

and both waves have their mean energy flux vector along $\boldsymbol{V}_3$.

For a combined wave solution $\boldsymbol{D} = c\boldsymbol{D}_3 + d\boldsymbol{D}_4$, $\boldsymbol{E} = c\boldsymbol{E}_3 + d\boldsymbol{E}_4$, etc., we find that the weighted energy density is given by

$$W = (2k_3 m_3)^{-1} (|c|^2 k_3 m_1 + |d|^2 k_1 m_3) \tag{9.57}$$

in accordance with (5.22), and that the weighted energy flux is given by

$$\boldsymbol{R} = |c|^2 \boldsymbol{R}_3 + |d|^2 \boldsymbol{R}_4 \pm \tfrac{1}{2} (c\bar{d})^- \alpha (k_1 m_1)^{\frac{1}{2}} (k_3^{-1} m_2^{-1} + k_2^{-1} m_3^{-1}) \, \boldsymbol{V}_1. \tag{9.58}$$

The interaction term in $\boldsymbol{R}$ is orthogonal to $\boldsymbol{V}_*^2$ and $\boldsymbol{V}_*^3$ in accordance with (5.26). Remarks analogous to those of possibility (9.45) may be easily formulated.

9.4. *Linearly polarized inhomogeneous waves*

We first note that if the amplitudes $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$, $\boldsymbol{H}$ are all linear bivectors, that is if all the fields **E**, **D**, **B**, **H** are linearly polarised, then the wave is necessarily homogeneous. Indeed, assuming $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$, $\boldsymbol{H}$ to be linear bivectors, the orthogonality relations (4.2) imply that $\boldsymbol{D}$ and $\boldsymbol{B}$ have different directions, and $\boldsymbol{C}\cdot\boldsymbol{D} = \boldsymbol{C}\cdot\boldsymbol{B} = 0$ show that $\boldsymbol{C}$ is along the direction orthogonal to the plane of $\boldsymbol{D}$ and $\boldsymbol{B}$. Thus $\boldsymbol{C}$ is a linear bivector, which means that the wave is homogeneous. We disregard this case here because homogeneous waves have been studied in §§7 and 8.

Let us now assume that $\boldsymbol{E}$, $\boldsymbol{D}$ are linear bivectors, whereas $\boldsymbol{H}$, $\boldsymbol{B}$ are not linear bivectors. Then the orthogonality relations (4.2) imply that both $\kappa^{-1}\boldsymbol{D}$ and $\mu^{-1}\boldsymbol{D}$ are orthogonal to the plane of the bivector $\boldsymbol{B}$. Thus for some scalar λ we have $(\mu^{-1}-\lambda\kappa^{-1})\,\boldsymbol{D}=0$, which shows that $\boldsymbol{D}$ must be along one of the eigenvectors $\boldsymbol{V}_1$, $\boldsymbol{V}_2$, $\boldsymbol{V}_3$ of the matrix μ^{-1} with respect to the metric κ^{-1}. Then, $\boldsymbol{C}\cdot\boldsymbol{D}=0$ implies that one of the components C_1, C_2, C_3 of the bivector $\boldsymbol{C}$ must be zero.

Analogously, assuming that $\boldsymbol{H}$, $\boldsymbol{B}$ are linear bivectors while $\boldsymbol{E}$, $\boldsymbol{D}$ are not linear bivectors, we arrive at the conclusion that one of the components C_1, C_2, C_3 of the bivector $\boldsymbol{C}$ must be zero.

We thus study the inhomogeneous waves corresponding to a bivector $\boldsymbol{C}$ with one zero component in the basis $\boldsymbol{V}^i_*$.

Let us assume $C_1=0$. Then the systems (9.11) and (9.12) for the determination of the eigenbivectors $\boldsymbol{E}$, $\boldsymbol{H}$ reduce to

$$\left.\begin{aligned}(N^{-2}k_1\,m_2\,m_3-m_2\,C_2^2-m_3\,C_3^2)\,E_1&=0,\\ (N^{-2}k_2\,m_1-C_3^2)\,E_2+C_2\,C_3\,E_3&=0,\\ C_3\,C_2\,E_2+(N^{-2}k_3\,m_1-C_2^2)\,E_3&=0,\end{aligned}\right\}\tag{9.59}$$

and

$$\left.\begin{aligned}(N^{-2}m_1\,k_2\,k_3-k_2\,C_2^2-k_3\,C_3^2)\,H_1&=0,\\ (N^{-2}m_2\,k_1-C_3^2)\,H_2+C_2\,C_3\,H_3&=0,\\ C_3\,C_2\,H_2+(N^{-2}m_3\,k_1-C_2^2)\,H_3&=0.\end{aligned}\right\}\tag{9.60}$$

The equations (9.59) and (9.60) yield non-trivial solutions for $\boldsymbol{E}$ and $\boldsymbol{H}$ provided N^{-2} is given either by

$$N^{-2}=k_1^{-1}(m_3^{-1}C_2^2+m_2^{-1}C_3^2)=(\det\mu)^{-1}\lambda_1^{-1}\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C},\tag{9.61}$$

or

$$N^{-2}=m_1^{-1}(k_3^{-1}C_2^2+k_2^{-1}C_3^2)=(\det\kappa)^{-1}\lambda_1\,\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C}.\tag{9.62}$$

We assume that they are distinct and both non-zero, because the cases of double roots and of zero roots have been studied in the preceding sections.

With N given by (9.61), we find

$$\left.\begin{aligned}&\boldsymbol{D}=k_1\,\boldsymbol{V}_1,\quad \boldsymbol{B}=N(C_3\,\boldsymbol{V}_2-C_2\,\boldsymbol{V}_3),\\ &\boldsymbol{E}=\boldsymbol{V}^1_*,\quad \boldsymbol{H}=N(C_3\,m_2^{-1}\boldsymbol{V}^2_*-C_2\,m_3^{-1}\boldsymbol{V}^3_*),\\ &4W=k_1+|N|^2(m_2\,m_3)^{-1}(m_2|C_2|^2+m_3|C_3|^2)=k_1+m_1\,(\det\mu)^{-1}\boldsymbol{S}^{\mathrm{T}}\mu\bar{\boldsymbol{S}},\\ &2\boldsymbol{R}=(m_2\,m_3)^{-1}(m_2\,S_2^+\,\boldsymbol{V}_2+m_3\,S_3^+\,\boldsymbol{V}_3)=m_1\,(\det\mu)^{-1}\mu\boldsymbol{S}^+.\end{aligned}\right\}\tag{9.63}$$

The expressions in (9.63) for W and $\boldsymbol{R}$ may be read off from equations (5.15) and (5.13) respectively on noting that $\boldsymbol{E}^{\mathrm{T}}\kappa\bar{\boldsymbol{E}}=k_1$, $\boldsymbol{E}^{\mathrm{T}}\mu\bar{\boldsymbol{E}}=m_1$, $\boldsymbol{S}^{\mathrm{T}}\mu\bar{\boldsymbol{E}}=m_1\,\boldsymbol{S}\cdot\boldsymbol{V}_1=0$. For this wave, the fields **E**, **D** are linearly polarized along $\boldsymbol{V}^1_*$ and $\boldsymbol{V}_1$ respectively, whereas the fields **B** and **H** are elliptically polarized. The mean energy flux vector is along the normal to the μ-metric ellipsoid at the point where the radius in the direction of $\boldsymbol{S}^+$ intersects it.

With N given by (9.62), we find

$$\left.\begin{aligned} \boldsymbol{D} &= -N(C_3\boldsymbol{V}_2 - C_2\boldsymbol{V}_3), \quad \boldsymbol{B} = m_1\boldsymbol{V}_1, \\ \boldsymbol{E} &= -N(C_3 k_2^{-1}\boldsymbol{V}_*^2 - C_2 k_3^{-1}\boldsymbol{V}_*^3), \quad \boldsymbol{H} = \boldsymbol{V}_*^1, \\ 4W &= m_1 + |N|^2(k_2k_3)^{-1}(k_2|C_2|^2 + k_3|C_3|^2) = m_1 + k_1(\det\kappa)^{-1}\boldsymbol{S}^{\mathrm{T}}\kappa\bar{\boldsymbol{S}}, \\ 2\boldsymbol{R} &= (k_2k_3)^{-1}(k_2S_2^+\boldsymbol{V}_2 + k_3S_3^+\boldsymbol{V}_3) = k_1(\det\kappa)^{-1}\kappa\boldsymbol{S}^+. \end{aligned}\right\} \tag{9.64}$$

The expressions in (9.64) for W and $\boldsymbol{R}$ may be read off from equations (5.16) and (5.14) respectively on noting that $\boldsymbol{H}^{\mathrm{T}}\mu\bar{\boldsymbol{H}} = m_1$, $\boldsymbol{H}^{\mathrm{T}}\kappa\bar{\boldsymbol{H}} = k_1$, $\boldsymbol{S}^{\mathrm{T}}\kappa\bar{\boldsymbol{H}} = k_1\boldsymbol{S}\cdot\boldsymbol{V}_1 = 0$. Here the fields **H**, **B** are linearly polarized along $\boldsymbol{V}_*^1$ and $\boldsymbol{V}_1$ respectively, whereas the fields **E** and **D** are elliptically polarized. The mean energy flux vector is along the normal to the κ-metric ellipsoid at the point where the radius in the direction of $\boldsymbol{S}^+$ intersects it.

Finally we note that similar results may be obtained in the case when C_2 or C_3 is zero.

10. Uniaxial crystals

Now the propagation of inhomogeneous plane waves in uniaxial crystals is investigated in detail.

In §10.1 we note that the secular equation may be factored. In the case of homogeneous waves the roots for N^{-2} correspond to the slowness surfaces, which are ellipsoids. Referring $\boldsymbol{C}$, $\boldsymbol{E}$ and $\boldsymbol{H}$ to the basis $\boldsymbol{V}_*^i$ and $\boldsymbol{D}$, $\boldsymbol{B}$ to the basis $\boldsymbol{V}_i$, the amplitudes are obtained for each mode, assuming that the two roots N^{-2} are distinct and non zero.

In §10.2 the possibility of the secular equation (3.15) or (3.16) for N^{-2} having zero roots is considered. It is shown that this equation has a double zero root (no wave propagation) when the ellipse of $\boldsymbol{C}$ is similar, and similarly situated, to the sections of the κ and μ-metric ellipsoid by the plane conjugate to the optic axis with respect to these ellipsoids. These sections, which are both similar and similarly situated, are called critical sections. Next it is shown that the secular equation has only one zero root (one propagation mode) when the ellipse of $\boldsymbol{C}$ is similar, and similarly situated, to an elliptical section of the κ or μ-metric ellipsoid other than the critical section.

Next, in §10.3, the possibility of a double root N^{-2} for the secular equation is considered. It is shown that, if $\boldsymbol{C}$ is not along the generalized optic axis (that is if the wave is not homogeneous), the amplitudes $\boldsymbol{D}$, $\boldsymbol{B}$, $\boldsymbol{E}$, $\boldsymbol{H}$ corresponding to this double root are determined up to an arbitrary scalar factor. In particular, $\boldsymbol{D}$ and $\boldsymbol{B}$ are parallel and the ellipses of $\boldsymbol{D}$ and $\boldsymbol{B}$ are both similar, and similarly situated, to either of the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane orthogonal to the generalized optic axis. The ellipses of $\boldsymbol{E}$ and $\boldsymbol{H}$ are both similar, and similarly situated, to the critical sections of the κ and μ-metric ellipsoids.

Finally, in §10.4, the possibility of inhomogeneous waves with **E** and **D**, or **B** and **H** linearly polarised is considered. It is shown that these waves may propagate either when the ellipse of $\boldsymbol{C}$ is in the plane conjugate to the generalized optic axis with respect to the κ and μ-metric ellipsoid, or when the ellipse of $\boldsymbol{C}$ is in a plane passing through the generalized optic axis.

10.1. *Propagation condition*

We here assume that
$$\lambda_1 > \lambda_2 = \lambda_3. \tag{10.1}$$
The case $\lambda_1 = \lambda_2 > \lambda_3$ may be dealt with in a similar way.

By specializing (9.8) or (9.9) we note that the secular equation for N^{-2} is now factored:
$$\{N^{-2} - \lambda_2^{-1}(\det\mu)^{-1}\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C}\}\{N^{-2} - \lambda_2(\det\kappa)^{-1}\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C}\} = 0. \tag{10.2}$$
The solutions are thus
$$N^{-2} = \lambda_2^{-1}(\det\mu)^{-1}\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C}, \tag{10.3}$$
and
$$N^{-2} = \lambda_2(\det\kappa)^{-1}\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C}. \tag{10.4}$$
If $\boldsymbol{C}$ is a real unit vector $\hat{\boldsymbol{n}}$ (homogeneous waves) the slowness surfaces corresponding to (10.3) and (10.4) are both ellipsoids. For want of a better terminology we relate the names of the two waves to the forms these surfaces would have if the crystal is magnetically isotropic. Thus with $\mu_{ij} = \mu\delta_{ij}$, the ellipsoid related to (10.3) is a sphere and hence we call the wave solution corresponding to (10.3) an 'ordinary wave'. Similarly, if $\mu_{ij} = \mu\delta_{ij}$, the ellipsoid related to (10.4) becomes a spheroid and the wave solution corresponding to (10.4) is called an 'extraordinary wave'.

Throughout this section we assume that no two of the components C_i are zero. For if this is so the wave is homogeneous, a case that has been considered in §7.

10.1.1. *Ordinary wave*

Introducing (10.3) into the propagation condition (3.10) for $\boldsymbol{E}$, and referring $\boldsymbol{C}$ and $\boldsymbol{E}$ to the basis $\boldsymbol{V}_*^i$, we obtain
$$\left.\begin{aligned} m_1 C_1(\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{E}) - (m_1 - \lambda_2^{-1}k_1)(\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C})E_1 = 0, \\ m_2 C_2(\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{E}) = 0, \quad m_3 C_3(\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{E}) \doteq 0. \end{aligned}\right\} \tag{10.5}$$
From this and from (3.2), (3.1*c*) we obtain the amplitudes $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$, $\boldsymbol{H}$ of the ordinary wave. We find
$$\left.\begin{aligned} \boldsymbol{E} &= m_1 m_3 C_3 \boldsymbol{V}_*^2 - m_1 m_2 C_2 \boldsymbol{V}_*^3, \\ \boldsymbol{D} &= \lambda_2(\det\mu)(C_3\boldsymbol{V}_2 - C_2\boldsymbol{V}_3), \\ \boldsymbol{B} &= N\{-m_1(m_2C_2^2 + m_3C_3^2)\boldsymbol{V}_1 + m_1m_2C_1C_2\boldsymbol{V}_2 + m_1m_3C_1C_3\boldsymbol{V}_3\}, \\ \boldsymbol{H} &= N\{-(m_2C_2^2 + m_3C_3^2)\boldsymbol{V}_*^1 + m_1C_1C_2\boldsymbol{V}_*^2 + m_1C_1C_3\boldsymbol{V}_*^3\}, \\ 2W &= m_1(m_2|C_2|^2 + m_3|C_3|^2)\boldsymbol{S}^{+\mathrm{T}}\mu\boldsymbol{S}^+ + 2\det\mu\{|N|(C_2\bar{C}_3)^-\}^2, \\ 2\boldsymbol{R} &= m_1(m_2|C_2|^2 + m_3|C_3|^2)\mu S^+ + 2\det\mu(C_2\bar{C}_3)^-(S_3^-\boldsymbol{V}_2 - S_2^-\boldsymbol{V}_3). \end{aligned}\right\} \tag{10.6}$$
The expressions in (10.6) for W and $\boldsymbol{R}$ may be read off from (5.19) and (5.13) respectively, on noting that $\boldsymbol{S}^{\mathrm{T}}\mu\bar{\boldsymbol{E}} = 2\mathrm{i}N\det\mu(C_2\bar{C}_3)^-$, $\boldsymbol{E}^{\mathrm{T}}\mu\boldsymbol{S}^+ = -\mathrm{i}\bar{N}\det\mu(C_2\bar{C}_3)^-$, and that $\boldsymbol{S}^{-\mathrm{T}}\mu\boldsymbol{S}^+ = 0$ as a consequence of (10.3). Also, using $\boldsymbol{S}^{-\mathrm{T}}\mu\boldsymbol{S}^+ = 0$, we recover easily (5.6).

10.1.2. *Extraordinary wave*

Introducing (10.4) into the propagation condition (3.11) for $\boldsymbol{H}$, and referring $\boldsymbol{C}$ and $\boldsymbol{H}$ to the basis $\boldsymbol{V}_*^i$, we obtain
$$\left.\begin{aligned} k_1 C_1(\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{H}) - (k_1 - \lambda_2 m_1)(\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C})H_1 = 0, \\ k_2 C_2(\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{H}) = 0, \quad k_3 C_3(\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{H}) = 0. \end{aligned}\right\} \tag{10.7}$$

From this and from (3.2), (3.1 *d*), we obtain the amplitudes $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$, $\boldsymbol{H}$ of the extraordinary wave. We find

$$\left.\begin{aligned} \boldsymbol{E} &= N\{(k_2 C_2^2 + k_3 C_3^2)\, \boldsymbol{V}_*^1 - k_1 C_1 C_2 \boldsymbol{V}_*^2 - k_1 C_1 C_3 \boldsymbol{V}_*^3\}, \\ \boldsymbol{D} &= N\{k_1(k_2 C_2^2 + k_3 C_3^2)\, \boldsymbol{V}_1 - k_1 k_2 C_1 C_2 \boldsymbol{V}_2 - k_1 k_3 C_1 C_3 \boldsymbol{V}_3\}, \\ \boldsymbol{B} &= \lambda_2^{-1} \det\kappa (C_3 \boldsymbol{V}_2 - C_2 \boldsymbol{V}_3), \quad \boldsymbol{H} = k_1 k_3 C_3 \boldsymbol{V}_*^2 - k_1 k_2 C_2 \boldsymbol{V}_*^3, \\ 2W &= k_1(k_2|C_2|^2 + k_3|C_3|^2)\, \boldsymbol{S}^{+\mathrm{T}} \kappa \boldsymbol{S}^+ + 2\det\kappa \{|N|\,(C_2 \bar{C}_3)^-\}^2, \\ 2\boldsymbol{R} &= k_1(k_2|C_2|^2 + k_3|C_3|^2)\, \kappa \boldsymbol{S}^+ + 2\det\kappa (C_2 \bar{C}_3)^- (S_3^- \boldsymbol{V}_2 - S_2^- \boldsymbol{V}_3). \end{aligned}\right\} \quad (10.8)$$

The expressions in (10.8) for W and $\boldsymbol{R}$ may be read off from (5.20) and (5.14) respectively on noting that $\boldsymbol{S}^{\mathrm{T}}\kappa\bar{\boldsymbol{H}} = 2\mathrm{i}N \det\kappa (C_2\bar{C}_3)^-$, that $\boldsymbol{H}^{\mathrm{T}}\kappa\boldsymbol{S}^+ = -\mathrm{i}\bar{N}\det\kappa(C_2\bar{C}_3)^-$, and that $\boldsymbol{S}^{-\mathrm{T}}\kappa\boldsymbol{S}^+ = 0$ as a consequence of (10.4). Also, using $\boldsymbol{S}^{-\mathrm{T}}\kappa\boldsymbol{S}^+ = 0$, we recover (5.6).

10.2. *Zero roots*

Here we seek the bivectors $\boldsymbol{C}$ such that the secular equation (10.2) has two zero roots (no propagation), or one zero root (only one mode of propagation).

10.2.1. *Both roots zero, no propagation*

The two roots (10.3) and (10.4) are zero when $\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C} = \boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C} = 0$. Setting to zero the right-hand sides of (9.2 *a*, *b*) with (10.1), we find that when $\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C} = \boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C} = 0$ the bivector $\boldsymbol{C}$ is given, up to a scalar factor, by $\boldsymbol{C} = \boldsymbol{C}_0$ or $\boldsymbol{C} = \bar{\boldsymbol{C}}_0$, where

$$\boldsymbol{C}_0 = (\mathrm{i}(k_3/k_2)^{\frac{1}{2}}\, \boldsymbol{V}_*^2 + \boldsymbol{V}_*^3) = (\mathrm{i}(m_3/m_2)^{\frac{1}{2}}\, \boldsymbol{V}_*^2 + \boldsymbol{V}_*^3). \quad (10.9)$$

Then using (8.31), we note

$$\boldsymbol{n}_0^{\mathrm{T}}\kappa\boldsymbol{C}_0 = \boldsymbol{n}_0^{\mathrm{T}}\mu\boldsymbol{C}_0 = 0, \quad (10.10)$$

and from the definition (8.29) of $\boldsymbol{A}_0$, we find

$$\boldsymbol{A}_0 \cdot \boldsymbol{C}_0 = 0. \quad (10.11)$$

We also have

$$\kappa\boldsymbol{C}_0 = (k_2 k_3)^{\frac{1}{2}}\boldsymbol{A}_0, \quad \mu\boldsymbol{C}_0 = (m_2 m_3)^{\frac{1}{2}}\boldsymbol{A}_0. \quad (10.12)$$

By using the Appendix B, we note that the ellipse of the bivector $\boldsymbol{C}_0$ is similar, and similarly situated, to either of the elliptical sections of the κ and μ-metric ellipsoids by the plane conjugate to the generalized optic axis with respect to either of these ellipsoids (plane spanned by the vectors $\boldsymbol{V}_*^2$, $\boldsymbol{V}_*^3$). The section of the κ or μ-metric ellipsoid by this plane will be called a critical section.

Thus, in a uniaxial crystal, no wave propagation is possible with a slowness bivector $\boldsymbol{S} = N\boldsymbol{C}$ whose ellipse is similar and similarly situated to the critical section of the κ or μ-metric ellipsoid, that is with a slowness bivector parallel to the bivector $\boldsymbol{C}_0$ defined by (10.9).

Remark. In the case of a magnetically (electrically) isotropic crystal, $\mu = \mu 1$ ($\kappa = \kappa 1$), and thus $\boldsymbol{C}_0$ is an isotropic eigenbivector of $\kappa(\mu)$, and the optic axis $\boldsymbol{n}_0$ is orthogonal to its plane. Thus no wave propagation is possible with a slowness bivector which is isotropic in the plane orthogonal to the optic axis. For the case of magnetically isotropic crystals, see also Hayes (1987, §7.4).

10.2.2. *One zero root. One propagation mode*

The equation (10.2) has one zero root when $\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C} = 0$ and $\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C} \neq 0$ (no extraordinary wave), or when $\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C} = 0$ and $\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C} \neq 0$ (no ordinary wave).

Case 1: $\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C} = 0$, $\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C} \neq 0$. *No extraordinary wave.* This case occurs when the ellipse of the bivector $\boldsymbol{C}$ is similar and similarly situated to any elliptical section of the κ-metric ellipsoid other than the critical section. Then the root (10.4) of the secular equation is zero, whereas the root (10.3) reduces to

$$N^{-2} = -k_2^{-1}k_3^{-1}(\lambda_1-\lambda_2)\,C_1^2, \tag{10.13}$$

which may be obtained as a special case of (9.23).

The corresponding field quantities are given as in the biaxial case by (9.24), or by (10.6).

Case 2: $\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C} = 0$, $\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C} \neq 0$. *No ordinary wave.* This case occurs when the ellipse of the bivector $\boldsymbol{C}$ is similar, and similarly situated, to any elliptical section of the μ-metric ellipsoid other than the critical section. Then the root (10.3) of the secular equation is zero, while the root (10.4) reduces to

$$N^{-2} = m_2^{-1}m_3^{-1}(\lambda_2^{-1}-\lambda_1^{-1})\,C_1^2, \tag{10.14}$$

which may be obtained as a special case of (9.25).

The corresponding field quantities are given as in the biaxial case by (9.26), or by (10.8).

Remark: When the crystal is magnetically (electrically) isotropic, $\mu = \mu 1$ ($\kappa = \kappa 1$) and then, in case 2 (case 1), $\boldsymbol{C}$ is isotropic but not in the plane orthogonal to the optic axis. As is the biaxial case, we note that the corresponding single propagation mode is such that $\boldsymbol{D}$ ($\boldsymbol{B}$) and $\boldsymbol{S}$ are parallel and isotropic (see Hayes 1987, §7.4 for the case of magnetically isotropic crystals).

In case 1 (case 2), however, $\boldsymbol{C}$ is not isotropic. Indeed only one wave may propagate when the ellipse of $\boldsymbol{C}$ is similar and similarly situated to any section of the κ-metric ellipsoid (μ-metric ellipsoid) other than the circular section.

10.3. *Double roots*: $\boldsymbol{D}$ *and* $\boldsymbol{B}$ *parallel*

The condition (9.28) or (9.29) for a double root reduces, owing to (10.1), to

$$k_2\,C_2^2 + k_3\,C_3^2 = 0, \tag{10.15}$$

or, equivalently,

$$m_2\,C_2^2 + m_3\,C_3^2 = 0, \tag{10.16}$$

that is

$$C_3 = \mp\mathrm{i}(k_2/k_3)^{\frac{1}{2}}\,C_2 = \mp\mathrm{i}(m_2/m_3)^{\frac{1}{2}}\,C_2. \tag{10.17}$$

We here assume that C_2 is different from zero, for otherwise $\boldsymbol{C}$ obeying (10.17) is along the generalized optic axis $\boldsymbol{V}_*^1$, and thus the corresponding wave is homogeneous. This case has been studied in §8.

Then, from either (10.3) or (10.4), the double eigenvalue $(-N^{-2})$ of the eigenvalue problems (3.4) and (3.6) is given by

$$N^{-2} = \lambda_2^{-1}m_2^{-1}m_3^{-1}C_1^2 = \lambda_2\,k_2^{-1}k_3^{-1}C_1^2, \tag{10.18}$$

and, assuming $C_1 \neq 0$ (the case of a double zero root has been considered in §10.2.1), it follows from (10.5) and (10.7) that there is a simple infinity of eigenbivectors $\boldsymbol{E}$ and $\boldsymbol{H}$ corresponding to this double eigenvalue.

The corresponding field quantities may be obtained by introducing (10.17) into (10.6) or (10.8). In this way we find, on choosing the upper $(-)$ sign in (10.17),

$$\left.\begin{aligned} \boldsymbol{D} &= k_3 m_2^{\frac{1}{2}} \boldsymbol{A}_0, \quad \boldsymbol{B} = \mathrm{i} m_3 k_2^{\frac{1}{2}} \boldsymbol{A}_0, \\ \boldsymbol{E} &= m_3^{\frac{1}{2}} \boldsymbol{C}_0, \quad \boldsymbol{H} = \mathrm{i} k_3^{\frac{1}{2}} \boldsymbol{C}_0, \\ W &= k_3 m_3, \quad \boldsymbol{R} = k_3 (m_3 k_2^{-1})^{\frac{1}{2}} \boldsymbol{V}_1, \end{aligned}\right\} \tag{10.19}$$

corresponding to the slowness

$$\boldsymbol{S} = (k_2 m_3)^{\frac{1}{2}} (\boldsymbol{V}_*^1 + C_1^{-1} C_2 \boldsymbol{V}_*^2 - \mathrm{i} (k_2 k_3^{-1})^{\frac{1}{2}} C_1^{-1} C_2 \boldsymbol{V}_*^3). \tag{10.20}$$

Thus, the ellipses of $\boldsymbol{D}$ and $\boldsymbol{B}$ are similar, and similarly situated, to either of the elliptical sections of the κ^{-1} and μ^{-1}-metric ellipsoids by the plane Π_0 orthogonal to the generalized optic axis $\boldsymbol{n}_0$ (plane spanned by the vectors $\boldsymbol{V}_2$, $\boldsymbol{V}_3$). The ellipses of $\boldsymbol{E}$ and $\boldsymbol{H}$ are both similar, and similarly situated, to either of the elliptical sections of the κ and the μ-metric ellipsoids by the plane conjugate to the generalized optic axis with respect to either of these ellipsoids (plane spanned by the vectors $\boldsymbol{V}_*^2$, $\boldsymbol{V}_*^3$), that is similar, and similarly situated, to the critical sections of the κ and μ-metric ellipsoids. Similarly, on choosing the lower $(+)$ sign (10.17) we find

$$\left.\begin{aligned} \boldsymbol{D} &= k_3 m_2^{\frac{1}{2}} \bar{\boldsymbol{A}}_0, \quad \boldsymbol{B} = -\mathrm{i} m_3 k_2^{\frac{1}{2}} \bar{\boldsymbol{A}}_0, \\ \boldsymbol{E} &= m_3^{\frac{1}{2}} \bar{\boldsymbol{C}}_0, \quad \boldsymbol{H} = -\mathrm{i} k_3^{\frac{1}{2}} \bar{\boldsymbol{C}}_0, \\ W &= k_3 m_3, \quad \boldsymbol{R} = k_3 (m_3 k_2^{-1})^{\frac{1}{2}} \boldsymbol{V}_1, \end{aligned}\right\} \tag{10.21}$$

corresponding to the slowness

$$\boldsymbol{S} = (k_2 m_3)^{\frac{1}{2}} (\boldsymbol{V}_*^1 + C_1^{-1} C_2 \boldsymbol{V}_*^2 + \mathrm{i} (k_2 k_3^{-1})^{\frac{1}{2}} C_1^{-1} C_2 \boldsymbol{V}_*^3). \tag{10.22}$$

The geometrical interpretation is the same as for (10.19) except that the ellipses of $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{H}$ have now the opposite handedness.

10.4. *Linearly polarized inhomogeneous waves*

As in §9.4, we note that if all the fields $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$, $\mathbf{H}$ are linearly polarized, the wave is homogeneous and we now disregard this case.

Let us now assume that $\boldsymbol{E}$, $\boldsymbol{D}$ are linear bivectors whereas $\boldsymbol{H}$, $\boldsymbol{B}$ are not linear bivectors. Thus, using the argument of §9.4, we have, for some scalar λ, $(\mu^{-1} - \lambda\kappa^{-1})\boldsymbol{D} = 0$, which here shows that $\boldsymbol{D}$ must be either along the eigenvector $\boldsymbol{V}_1$ of the matrix μ^{-1} with respect to the metric κ^{-1}, or in the plane spanned by the eigenvectors $\boldsymbol{V}_2$, $\boldsymbol{V}_3$ corresponding to the double eigenvalue $\lambda_2 = \lambda_3$. Then, $\boldsymbol{C} \cdot \boldsymbol{D} = 0$ implies that either $C_1 = 0$ or that the ratio of the components C_2, C_3 is real.

Let us first assume $C_1 = 0$. The roots (10.3) and (10.4) of the secular equation corresponding respectively to the ordinary and extraordinary wave then become

$$N^{-2} = \lambda_2^{-1} (\det \mu)^{-1} \boldsymbol{C}^{\mathrm{T}} \mu \boldsymbol{C} = m_1^{-1} (k_3^{-1} C_2^2 + k_2^{-1} C_3^2) = \lambda_1 (\det \kappa)^{-1} \boldsymbol{C}^{\mathrm{T}} \kappa \boldsymbol{C} \tag{10.23}$$

(ordinary wave) and

$$N^{-2} = \lambda_2 (\det \kappa)^{-1} \boldsymbol{C}^{\mathrm{T}} \kappa \boldsymbol{C} = k_1^{-1} (m_3^{-1} C_2^2 + m_2^{-1} C_3^2) = \lambda_1^{-1} (\det \mu)^{-1} \boldsymbol{C}^{\mathrm{T}} \mu \boldsymbol{C} \tag{10.24}$$

(extraordinary wave).

The field quantities (10.6) associated with the ordinary wave then reduce up to a scalar factor to (9.64), so that the fields **H** and **B** of this wave are linearly polarized along V^1_* and V_1 respectively, whereas the fields **E** and **D** are elliptically polarized.

The field quantities (10.8) associated with the extraordinary wave reduce up to a scalar factor to (9.63), so that the fields **E** and **D** of this wave are linearly polarized along V^1_* and V_1 respectively, whereas the fields **H** and **B** are elliptically polarized.

Next we assume that $C_1 \neq 0$ and that the ratio of the components C_2, C_3 of the bivector $\boldsymbol{C}$ is real. This includes the cases when C_2 or C_3 is zero. Then, the ellipse of the bivector $\boldsymbol{C}$ lies in a plane passing through the vector $\boldsymbol{n}_0 = V^1_*$, that is a plane passing through the generalized optic axis. Now (10.6) shows that for the ordinary wave the fields **E** and **D** are linearly polarized in the plane spanned respectively by the vectors V^2_*, V^3_* and V_2, V_3 (plane Π_0), whereas the fields **H** and **B** are elliptically polarized. Also (10.8) shows that for the extraordinary wave the fields **H** and **B** are linearly polarized in the plane spanned respectively by the vectors V^2_*, V^3_* and V_2, V_3 (plane Π_0), whereas the fields **E** and **D** are elliptically polarized. Moreover, because the ratio of the components C_2, C_3 is real, we have $(C_2 \bar{C}_3)^- = 0$ and the mean energy flux of the ordinary wave lies along $\mu \boldsymbol{S}^+$ while the mean energy flux of the extraordinary wave lies along $\kappa \boldsymbol{S}^+$.

11. Pseudo-isotropic crystals

Now we consider the case of pseudo-isotropic crystals. Then the electric permittivity tensor κ is a scalar multiple of the magnetic permeability tensor μ.

In §11.1 we note that the secular equation has in this case a double root N^{-2} for every bivector $\boldsymbol{C}$. In the case of homogeneous waves this means that the two slowness surfaces associated with a uniaxial crystal coalesce into a single ellipsoid. The propagation condition then shows that $\boldsymbol{D}$ or $\boldsymbol{B}$ may be chosen to be any bivector orthogonal to the bivector $\boldsymbol{C}$. There is thus a double infinity of eigenbivectors $\boldsymbol{E}$ or $\boldsymbol{H}$ corresponding to the double root of the secular equation.

In §11.2 we note that the double root of the secular equation is zero (no wave propagation) when the ellipse of $\boldsymbol{C}$ is similar and similarly situated to the section of the κ and μ-metric ellipsoids by any plane. All these sections are called *critical sections*.

Next, in §11.3, we show that for every bivector $\boldsymbol{C}$ (not linear) there is an inhomogeneous wave with **E** and $\boldsymbol{D}$ linearly polarized, and another one with **H** and **B** linearly polarized.

In §11.4, these two waves are combined to form the general inhomogeneous wave solution.

Finally, in §11.5, we show that for every bivector $\boldsymbol{C}$, there are two waves for which the bivectors $\boldsymbol{D}$ and $\boldsymbol{B}$ are parallel. The ellipses of $\boldsymbol{D}$ and $\boldsymbol{B}$ are both similar, and similarly situated, to a section of the κ^{-1} or μ^{-1}-metric ellipsoid. Also, the bivectors $\boldsymbol{E}$ and $\boldsymbol{H}$ are parallel and their ellipses are both similar and similarly situated to a critical section of the κ or μ-metric ellipsoid.

11.1. *Propagation condition*

We here assume that

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda, \tag{11.1}$$

that is

$$\kappa = \lambda\mu. \tag{11.2}$$

The κ and μ-metric ellipsoids are similar, and similarly situated, and so are the κ^{-1}- and μ^{-1}-

metric ellipsoids. For actual isotropic media all these ellipsoids are spheres. Introducing (11.2) into (3.15) and (3.16) we note that the secular equation then reduces to

$$\{N^{-2}-\lambda^{-1}(\det\mu)^{-1}(\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C})\}^2=0, \tag{11.3}$$

or, equivalently, to

$$\{N^{-2}-\lambda(\det\kappa)^{-1}(\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C})\}^2=0. \tag{11.4}$$

This equation has thus always the double root

$$N^{-2}=\lambda^{-1}(\det\mu)^{-1}(\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C})=\lambda(\det\kappa)^{-1}(\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C}). \tag{11.5}$$

Introducing (11.2) and (11.5) into the propagation conditions (3.10) and (3.11) for $\boldsymbol{E}$ and $\boldsymbol{H}$, we note that they reduce to

$$\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{E}=0, \quad \text{or} \quad \boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{E}=0, \tag{11.6}$$

and

$$\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{H}=0, \quad \text{or} \quad \boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{H}=0. \tag{11.7}$$

Thus, recalling (3.2), $\boldsymbol{D}$ ($\boldsymbol{B}$) may be any bivector orthogonal to the bivector $\boldsymbol{C}$. Then, $\boldsymbol{E}$ ($\boldsymbol{H}$), $\boldsymbol{B}$ ($\boldsymbol{D}$), $\boldsymbol{H}$ ($\boldsymbol{E}$) are determined from (3.1 *c*, *d*).

11.2. *Zero roots*

The double root (11.5) of the secular equation is zero when the bivector $\boldsymbol{C}$ is such that $\boldsymbol{C}^{\mathrm{T}}\kappa\boldsymbol{C}=\boldsymbol{C}^{\mathrm{T}}\mu\boldsymbol{C}=0$. Thus, in a pseudo-isotropic crystal, no wave propagation is possible with a slowness bivector $\boldsymbol{S}=N\boldsymbol{C}$ whose ellipse is similar, and similarly situated, to any section of the κ and μ-metric ellipsoids. All these sections will be called critical sections.

Remarks. In the case of isotropic crystals, $\mu=\mu 1$ and $\kappa=\kappa 1$, no wave propagation is possible with an isotropic slowness bivector (see Hayes 1987).

11.3. *Linearly polarized inhomogeneous waves*

For any bivector $\boldsymbol{C}$ whose ellipse is not similar and similarly situated to a critical section, $\boldsymbol{D}$ or $\boldsymbol{B}$ may be chosen to be any bivector orthogonal to the bivector $\boldsymbol{C}$. Let us here assume that $\boldsymbol{C}$ is not a linear bivector since the case of homogeneous waves has been considered in §§7 and 8. Then, $\boldsymbol{D}$ or $\boldsymbol{B}$ may be in particular chosen to be a linear bivector in the direction orthogonal to the plane of $\boldsymbol{C}=m\hat{\boldsymbol{m}}+\mathrm{i}\hat{\boldsymbol{n}}$, that is in the direction $\hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}}$.

We denote $\boldsymbol{D}$ by $\boldsymbol{D}_1$ when it is in the direction $\hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}}$. Then the corresponding field quantities $\boldsymbol{D}_1$, $\boldsymbol{E}_1$, etc..., are

$$\left.\begin{aligned}
\boldsymbol{D}_1&=\hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}}, \quad \boldsymbol{E}_1=(\det\kappa^{-1})\,\kappa\hat{\boldsymbol{m}}\times\kappa\hat{\boldsymbol{n}},\\
\boldsymbol{B}_1&=N(\det\kappa^{-1})\,\kappa\{\hat{\boldsymbol{m}}(\hat{\boldsymbol{n}}^{\mathrm{T}}\kappa\boldsymbol{C})-\hat{\boldsymbol{n}}(\hat{\boldsymbol{m}}^{\mathrm{T}}\kappa\boldsymbol{C})\},\\
\boldsymbol{H}_1&=N(\det\kappa^{-1})\,\lambda\{\hat{\boldsymbol{m}}(\hat{\boldsymbol{n}}^{\mathrm{T}}\kappa\boldsymbol{C})-\hat{\boldsymbol{n}}(\hat{\boldsymbol{m}}^{\mathrm{T}}\kappa\boldsymbol{C})\},\\
2W_1&=\lambda(\det\kappa^{-1})\{(\hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}})^{\mathrm{T}}\kappa^{-1}(\hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}})\}\,\boldsymbol{S}^{+\mathrm{T}}\kappa\boldsymbol{S}^{+},\\
2\boldsymbol{R}_1&=\lambda(\det\kappa^{-1})\{(\hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}})^{\mathrm{T}}\kappa^{-1}(\hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}})\}\,\kappa\boldsymbol{S}^{+}.
\end{aligned}\right\} \tag{11.8}$$

In deriving (11.8) use has been made of the identities (A 1), (A 2) and (A 3) of Appendix A. The expressions in (11.8) for W_1 and $\boldsymbol{R}_1$ may be read off from (5.19) and (5.13) respectively on noting that $\boldsymbol{S}^{\mathrm{T}}\mu\bar{\boldsymbol{E}}_1=\lambda^{-1}\boldsymbol{S}\cdot\bar{\boldsymbol{D}}_1=0$, and that $\boldsymbol{S}^{-\mathrm{T}}\kappa\boldsymbol{S}^{+}=\boldsymbol{S}^{-\mathrm{T}}\mu\boldsymbol{S}^{+}=0$ as a consequence of (11.5). Also, (5.6) may be easily checked.

The fields $\mathbf{E}_1$ and $\mathbf{D}_1$ are linearly polarized while the fields $\mathbf{H}_1$ and $\mathbf{B}_1$ are elliptically polarized. Also, from (11.8), we note that the bivector $\boldsymbol{H}_1$ is in the plane of the bivector $\boldsymbol{C}$. Thus, from (11.7) we conclude, using Appendix B, that the ellipse of $\boldsymbol{H}_1$ is similar, and similarly situated, to the polar reciprocal of the ellipse of $\boldsymbol{C}$ with respect to either of the elliptical sections of the κ and μ-metric ellipsoids by the plane of $\boldsymbol{C}$.

Next we denote $\boldsymbol{B}$ by $\boldsymbol{B}_2$ when it is in the direction $\hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}}$. Then the corresponding field quantities $\boldsymbol{D}_2$, $\boldsymbol{E}_2$, etc., are

$$\left.\begin{aligned}\boldsymbol{B}_2 &= \hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}}, \quad \boldsymbol{H}_2 = (\det\mu)^{-1}\mu\hat{\boldsymbol{m}}\times\mu\hat{\boldsymbol{n}},\\ \boldsymbol{D}_2 &= -N(\det\mu^{-1})\,\mu\{\hat{\boldsymbol{m}}(\hat{\boldsymbol{n}}^{\mathrm{T}}\mu\boldsymbol{C})-\hat{\boldsymbol{n}}(\hat{\boldsymbol{m}}^{\mathrm{T}}\mu\boldsymbol{C})\},\\ \boldsymbol{E}_2 &= -N(\det\mu^{-1})\,\lambda^{-1}\{\hat{\boldsymbol{m}}(\hat{\boldsymbol{n}}^{\mathrm{T}}\mu\boldsymbol{C})-\hat{\boldsymbol{n}}(\hat{\boldsymbol{m}}^{\mathrm{T}}\mu\boldsymbol{C})\},\\ 2W_2 &= \lambda^{-1}(\det\mu^{-1})\{(\hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}})^{\mathrm{T}}\mu^{-1}(\hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}})\}\,\boldsymbol{S}^{+\mathrm{T}}\mu\boldsymbol{S}^{+},\\ 2\boldsymbol{R}_2 &= \lambda^{-1}(\det\mu^{-1})\{(\hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}})^{\mathrm{T}}\mu^{-1}(\hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}})\}\,\mu\boldsymbol{S}^{+}.\end{aligned}\right\} \tag{11.9}$$

The expressions in (11.9) for W_2 and $\boldsymbol{R}_2$ may be read off from (5.20) and (5.14) respectively on noting that $\boldsymbol{S}^{\mathrm{T}}\kappa\bar{\boldsymbol{H}}_2 = \lambda\boldsymbol{S}\cdot\bar{\boldsymbol{B}}_2 = 0$, and that $\boldsymbol{S}^{-\mathrm{T}}\kappa\boldsymbol{S}^{+} = \boldsymbol{S}^{-\mathrm{T}}\mu\boldsymbol{S}^{+} = 0$. Also, (5.6) may be easily checked.

Now, the fields $\mathbf{H}_2$ and $\mathbf{B}_2$ are linearly polarized whereas the fields $\mathbf{E}_2$ and $\mathbf{D}_2$ are elliptically polarized. Also from (11.9), we note that the bivector $\boldsymbol{E}_2$ is in the plane of the bivector $\boldsymbol{C}$. Thus, from (11.6) we conclude, using Appendix B, that the ellipse of $\boldsymbol{E}_2$ is similar, and similarly situated, to the polar reciprocal of the ellipse of $\boldsymbol{C}$ with respect to either of the elliptical sections of the κ and μ-metric ellipsoids by the plane of $\boldsymbol{C}$.

11.4. *General inhomogeneous wave solution*

As there is a double infinity of eigenvectors $\boldsymbol{E}$ and $\boldsymbol{H}$ satisfying respectively the propagation conditions (11.6) and (11.7), any linear combination with possibility complex coefficients a and b of the amplitude of the waves (11.8) and (11.9) defines a wave (general inhomogeneous wave solution) with the same slowness $\boldsymbol{S} = N\boldsymbol{C}$. Also, it is easy to check that the amplitudes of the waves (11.8) and (11.9) satisfy the relations (4.15)–(4.20) so that these two wave solutions are those considered in §4.3 (case 2) and in §5.2.

For the general inhomogeneous wave solution $\boldsymbol{D} = a\boldsymbol{D}_1 + b\boldsymbol{D}_2$, $\boldsymbol{E} = a\boldsymbol{E}_1 + b\boldsymbol{E}_2$, etc., we find that the weighted energy density is given by

$$2W = \lambda(|a|^2+\lambda|b|^2)\,(\det\kappa^{-1})\,\{(\hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}})^{\mathrm{T}}\kappa^{-1}(\hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}})\}\,\boldsymbol{S}^{+\bar{\mathrm{T}}}\kappa\boldsymbol{S}^{+}, \tag{11.10}$$

in accordance with (5.22), and that the weighted energy flux is given by

$$\boldsymbol{R} = |a|^2\boldsymbol{R}_1 + |b|^2\boldsymbol{R}_2 + (a\bar{b})^{-}\lambda^2\,(\det\kappa^{-1})\,\{(\hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}})\,\kappa^{-1}(\hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}})\}\,\boldsymbol{S}^{+}\times\boldsymbol{S}^{-}, \tag{11.11}$$

in accordance with (5.27). This may be written

$$\boldsymbol{R} = l(|a|^2+\lambda|b|^2)\,\{\kappa\boldsymbol{S}^{+} + \sin 2\varphi\sin(\theta-\delta)\,\lambda^{\frac{1}{2}}\boldsymbol{S}^{+}\times\boldsymbol{S}^{-}\}, \tag{11.12}$$

where

$$\left.\begin{aligned} a = |a|\,\mathrm{e}^{\mathrm{i}\theta}, \quad b = |b|\,\mathrm{e}^{\mathrm{i}\delta}, \quad \mathrm{tg}\,\varphi = |a|/(\lambda^{\frac{1}{2}}|b|),\\ 2l = \lambda\,(\det\kappa^{-1})\,\{(\hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}})^{\mathrm{T}}\kappa^{-1}(\hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}})\}.\end{aligned}\right\} \tag{11.13}$$

Thus as $|a|$, $|b|$, θ and δ are varied, $\boldsymbol{R}$ varies from lying along $\kappa\boldsymbol{S}^{+}+\lambda^{\frac{1}{2}}\boldsymbol{S}^{+}\times\boldsymbol{S}^{-}$ to lying along $\kappa\boldsymbol{S}^{+}-\lambda^{\frac{1}{2}}\boldsymbol{S}^{+}\times\boldsymbol{S}^{-}$ and may take any intermediate direction between these two extremes.

11.5. *D and B parallel*

Among all the waves that may propagate in a pseudo-isotropic crystal, there are, for every bivector $\boldsymbol{C}$, two waves such that $\boldsymbol{D}$ and $\boldsymbol{B}$ are parallel.

Indeed, from (4.6), we conclude that $\boldsymbol{D}$ and $\boldsymbol{B}$ are parallel when σ given by (4.5) is zero. Then, computing $\boldsymbol{E}^{\mathrm{T}}\kappa\boldsymbol{E}$, or $\boldsymbol{H}^{\mathrm{T}}\mu\boldsymbol{H}$, or $\boldsymbol{D}^{\mathrm{T}}\kappa^{-1}\boldsymbol{D}$, or $\boldsymbol{B}^{\mathrm{T}}\mu^{-1}\boldsymbol{B}$ for the general inhomogeneous wave solution, we obtain

$$\sigma = (a^2+\lambda b^2)\,(\hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}})^{\mathrm{T}}\kappa^{-1}(\hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}}) = \lambda^{-1}(a^2+\lambda b^2)\,(\hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}})^{\mathrm{T}}\mu^{-1}(\hat{\boldsymbol{m}}\times\hat{\boldsymbol{n}}). \qquad (11.14)$$

Thus $\sigma = 0$, and hence $\boldsymbol{D}$ and $\boldsymbol{B}$ are parallel, when

$$a = \pm \mathrm{i}\lambda^{\frac{1}{2}}b. \qquad (11.15)$$

For these waves the ellipses of $\boldsymbol{D}$ and $\boldsymbol{B}$ are both similar, and similarly situated to an elliptical section of the κ^{-1} or μ^{-1}-metric ellipsoid. From (3.2) and (11.2), it is clear that $\boldsymbol{E}$ and $\boldsymbol{H}$ are also parallel. As $\sigma = 0$, the ellipses of $\boldsymbol{E}$ and $\boldsymbol{H}$ are both similar and similarly situated to an elliptical section of the κ or μ-metric ellipsoid (critical section). However, in general the plane of the bivectors $\boldsymbol{D}$ and $\boldsymbol{B}$ is not the same as the plane of the bivectors $\boldsymbol{E}$ and $\boldsymbol{H}$.

We are greatly indebted to X. Hubaut for valuable discussions concerning the geometry of ellipses and ellipsoids. We thank A. Gribaumont, X. Hubaut and E. Cox for their help with the diagrams and D. Heffernan for his advice concerning high temperature superconductors.

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Appendix A. Three identities

Let g be a real symmetric non-singular matrix. Let $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{T}$ be any four bivectors or vectors. Then we have the following identities

$$(\det g)\, g^{-1}(\boldsymbol{P}\times\boldsymbol{Q}) = g\boldsymbol{P}\times g\boldsymbol{Q}, \qquad (\mathrm{A}\ 1)$$

$$(\det g)\, g^{-1}\{\boldsymbol{P}\times g^{-1}(\boldsymbol{Q}\times\boldsymbol{R})\} = \boldsymbol{Q}(\boldsymbol{P}^{\mathrm{T}}g\boldsymbol{R}) - \boldsymbol{R}(\boldsymbol{P}^{\mathrm{T}}g\boldsymbol{Q}), \qquad (\mathrm{A}\ 2)$$

$$(\det g)\, (\boldsymbol{P}\times\boldsymbol{Q})^{\mathrm{T}}g^{-1}(\boldsymbol{R}\times\boldsymbol{T}) = (\boldsymbol{P}^{\mathrm{T}}g\boldsymbol{R})\,(\boldsymbol{Q}^{\mathrm{T}}g\boldsymbol{T}) - (\boldsymbol{P}^{\mathrm{T}}g\boldsymbol{T})\,(\boldsymbol{Q}^{\mathrm{T}}g\boldsymbol{R}). \qquad (\mathrm{A}\ 3)$$

For the identity (A 1) see for example Milne (1948). The identities (A 2) and (A 3) then follow by application of (A 1).

Appendix B. Orthogonality of bivectors with respect to a metric

Here we give a geometrical interpretation of the orthogonality of a pair of bivectors with respect to a metric g. Associated with each bivector is an ellipse. In part I we consider the case when these ellipses are coplanar. The orthogonality means that one of the ellipses is similar, and similarly situated, to the polar reciprocal of the other ellipse with respect to the section of the g-metric ellipsoid by the common plane of the two ellipses. In part II, the ellipses need not be coplanar. The plane of one ellipse may not contain the conjugate direction to the plane of the other ellipse with respect to the g-metric ellipsoid. Moreover, the g-projection of the first ellipse onto the plane of the second ellipse is similar, and similarly situated, to the polar reciprocal of the second ellipse with respect to the section of the g-metric ellipsoid by the plane of the second ellipse.

Let $\boldsymbol{P} = \boldsymbol{P}^+ + \mathrm{i}\boldsymbol{P}^-$, $\boldsymbol{Q} = \boldsymbol{Q}^+ + \mathrm{i}\boldsymbol{Q}^-$ be two bivectors orthogonal with respect to the real positive definite matrix g:

$$\boldsymbol{P}^{\mathrm{T}} g \boldsymbol{Q} = 0, \tag{B 1}$$

or equivalently,

$$\boldsymbol{P}^{+\mathrm{T}} g \boldsymbol{Q}^+ - \boldsymbol{P}^{-\mathrm{T}} g \boldsymbol{Q}^- = 0, \tag{B 2}$$

$$\boldsymbol{P}^{+\mathrm{T}} g \boldsymbol{Q}^- + \boldsymbol{P}^{-\mathrm{T}} g \boldsymbol{Q}^+ = 0. \tag{B 3}$$

With the bivectors $\boldsymbol{P}$, $\boldsymbol{Q}$ are associated directional ellipses (Hayes 1984) defined by the parametric equations

$$\boldsymbol{x} = \boldsymbol{P}^+ \cos\theta + \boldsymbol{P}^- \sin\theta, \quad \boldsymbol{x} = \boldsymbol{Q}^+ \cos\theta + \boldsymbol{Q}^- \sin\theta. \tag{B 4}$$

Part I: $\boldsymbol{P}$ and $\boldsymbol{Q}$ coplanar

To analyse (B 1) let us first assume that the ellipses associated with the bivectors $\boldsymbol{P}$, $\boldsymbol{Q}$ are in the same plane. If $\boldsymbol{x}$ denotes the position vector of a generic point of this plane, then

$$\boldsymbol{x}^{\mathrm{T}} g \boldsymbol{x} = 1 \tag{B 5}$$

is the equation of an ellipse. We call this the 'g-metric ellipse' or the 'metric ellipse' for brevity.

PROPERTY 1. *The ellipse of $\boldsymbol{Q}$ is similar, and similarly situated, to the polar reciprocal of the ellipse of $\boldsymbol{P}$ with respect to the metric ellipse. The ellipses of $\boldsymbol{P}$ and $\boldsymbol{Q}$ are described in the same sense (see figure 4).*

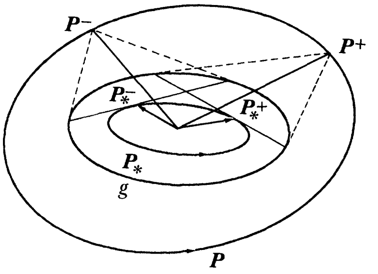

FIGURE 4. The ellipse of $\boldsymbol{P}_*$ is the polar reciprocal of the ellipse of $\boldsymbol{P}$ with respect to the ellipse g. The polar of $\boldsymbol{P}^+$ touches the ellipse of $\boldsymbol{P}_*$ at $\boldsymbol{P}_*^+$ and the polar of $\boldsymbol{P}^-$ touches the ellipse of $\boldsymbol{P}_*$ at $\boldsymbol{P}_*^-$. The ellipse of $\boldsymbol{Q}$ is similar, and similarly situated, to the ellipse of $\boldsymbol{P}_*$.

Proof. The polar reciprocal of the ellipse of $\boldsymbol{P}$ with respect to the metric ellipse is by definition the envelope of the polars of all the points of the ellipse of $\boldsymbol{P}$ with respect to the metric ellipse. Its parametric equation, with θ as parameter, may be obtained by solving for $\boldsymbol{x}$ the system

$$(\boldsymbol{P}^+ \cos\theta + \boldsymbol{P}^- \sin\theta)^{\mathrm{T}} g \boldsymbol{x} = 1, \quad (-\boldsymbol{P}^+ \sin\theta + \boldsymbol{P}^- \cos\theta)^{\mathrm{T}} g \boldsymbol{x} = 0. \tag{B 6a, b}$$

Indeed, (B 6a) is the equation of the polar of the point $\boldsymbol{P}^+ \cos\theta + \boldsymbol{P}^- \sin\theta$ with respect to the metric ellipse, and (B 6b) its derivative with respect to θ.

We introduce in the plane of $\boldsymbol{P}$ the set of vectors $\boldsymbol{P}_*^+$, $\boldsymbol{P}_*^-$ reciprocal to the set $\boldsymbol{P}^+$, $\boldsymbol{P}^-$ with respect to the metric g. Thus

$$\boldsymbol{P}^{-\mathrm{T}} g \boldsymbol{P}_*^+ = 0, \quad \boldsymbol{P}^{+\mathrm{T}} g \boldsymbol{P}_*^+ = 1, \tag{B 7}$$

and

$$\boldsymbol{P}^{+\mathrm{T}} g \boldsymbol{P}_*^- = 0, \quad \boldsymbol{P}^{-\mathrm{T}} g \boldsymbol{P}_*^- = 1. \tag{B 8}$$

The vector $\boldsymbol{P}_*^+$ is in the conjugate direction to the direction of $\boldsymbol{P}^-$ with respect to the metric ellipse and has its extremity on the polar of the extremity of $\boldsymbol{P}^+$ with respect to the metric ellipse. Similarly the vector $\boldsymbol{P}_*^-$ is in the conjugate direction to the direction of $\boldsymbol{P}^+$ and has its extremity on the polar of the extremity of $\boldsymbol{P}^-$ (see figure 4).

Then, it may be easily checked that the solution of (B 6) for $\boldsymbol{x}$ is given by

$$\boldsymbol{x} = \boldsymbol{P}_*^+ \cos\theta + \boldsymbol{P}_*^- \sin\theta, \tag{B 9}$$

so that the polar reciprocal of the ellipse of $\boldsymbol{P}$ with respect to the metric ellipse is the ellipse associated with the bivector $\boldsymbol{P}_* = \boldsymbol{P}_*^+ + \mathrm{i}\boldsymbol{P}_*^-$. It results from (B 7) and (B 8) that

$$\boldsymbol{P}^{\mathrm{T}} g \boldsymbol{P}_* = 0. \tag{B 10}$$

Then, because $\boldsymbol{P}_*$ and $\boldsymbol{Q}$ are both in the plane of $\boldsymbol{P}$, equation (B 10) together with (B 1), implies that $\boldsymbol{Q}$ and $\boldsymbol{P}_*$ are two parallel bivectors, which means (Hayes 1984) that the ellipse of $\boldsymbol{Q}$ is similar and similarly situated to the ellipse of $\boldsymbol{P}_*$.

Moreover, the pairs of vectors $(\boldsymbol{P}^+, \boldsymbol{P}^-)$ and $(\boldsymbol{Q}^+, \boldsymbol{Q}^-)$ have the same orientation, because

$$(\det g) \begin{vmatrix} \boldsymbol{P}^+ \\ \boldsymbol{P}^- \end{vmatrix} \begin{vmatrix} \boldsymbol{Q}^+ \\ \boldsymbol{Q}^- \end{vmatrix} = \begin{vmatrix} \boldsymbol{P}^{+\mathrm{T}} g \boldsymbol{Q}^+ & \boldsymbol{P}^{+\mathrm{T}} g \boldsymbol{Q}^- \\ \boldsymbol{P}^{-\mathrm{T}} g \boldsymbol{Q}^+ & \boldsymbol{P}^{-\mathrm{T}} g \boldsymbol{P}^- \end{vmatrix} = (\boldsymbol{P}^{+\mathrm{T}} g \boldsymbol{Q}^+)^2 + (\boldsymbol{P}^{+\mathrm{T}} g \boldsymbol{Q}^-)^2 > 0. \tag{B 11}$$

This means that the ellipses of $\boldsymbol{P}$ and $\boldsymbol{Q}$ are described in the same sense (Hayes 1984).

Remark 1. The polar reciprocal of any ellipse W with respect to an ellipse similar, and similarly situated, to W is again an ellipse similar, and similarly situated, to W.

Remark 2. Taking $\boldsymbol{Q} = \boldsymbol{P}$, we note that

$$\boldsymbol{P}^{\mathrm{T}} g \boldsymbol{P} = 0, \tag{B 12}$$

means that the ellipse of $\boldsymbol{P}$ is similar, and similarly situated, to its own polar reciprocal with respect to the metric ellipse. Then the bivector $\boldsymbol{P}$ is said to be isotropic with respect to the metric g. In this case, (B 2) and (B 3) reduce to

$$\boldsymbol{P}^{+\mathrm{T}} g \boldsymbol{P}^+ = \boldsymbol{P}^{-\mathrm{T}} g \boldsymbol{P}^-, \quad \boldsymbol{P}^{+\mathrm{T}} g \boldsymbol{P}^- = 0, \tag{B 13}$$

so that $\boldsymbol{P}^+$ and $\boldsymbol{P}^-$ are in conjugate directions of the metric ellipse, and their extremities are on an ellipse similar and similarly situated to the metric ellipse. Thus, (B 12) has the following geometrical interpretation: the ellipse of $\boldsymbol{P}$ is similar, and similarly situated, to the g-metric ellipse.

Part II: ***P*** *and* ***Q*** *not coplanar*

Let us now assume that the ellipses associated with the bivectors $\boldsymbol{P}$, $\boldsymbol{Q}$ are not necessarily in the same plane. In three-dimensional space, (B 5) is the equation of an ellipsoid. We call this the 'g-metric ellipsoid' or the 'metric ellipsoid' for brevity.

It is convenient to introduce here the concept of orthogonal projection with respect to the metric g, or g-projection. Let α be a given plane through the origin. Let $\boldsymbol{n}$ be a vector along the conjugate direction to α with respect to the metric ellipsoid. Then the equation of the plane α is

$$\boldsymbol{n}^{\mathrm{T}} g \boldsymbol{x} = 0. \tag{B 14}$$

It is easily seen that every vector $\boldsymbol{x}$ may be decomposed in a unique way as the sum of two

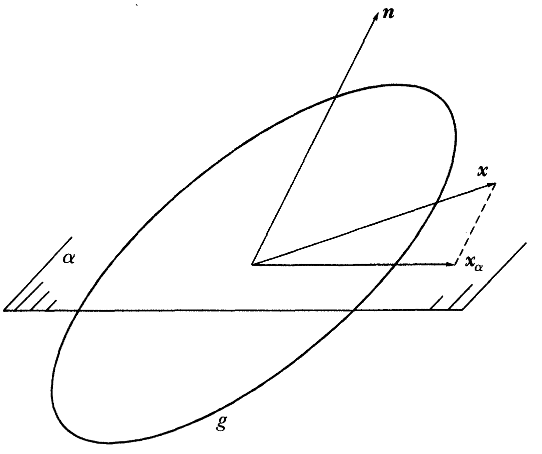

FIGURE 5. The plane α passes through the centre of the ellipsoid g. It is parallel to the tangent plane to g at the point where the vector $\boldsymbol{n}$ meets the ellipsoid. Any vector $\boldsymbol{x}$ may be decomposed into the sum of a component along $\boldsymbol{n}$ and a component in the plane α, which is called the g projection of $\boldsymbol{x}$ onto the plane α.

component vectors, one in the plane α, and the other parallel to $\boldsymbol{n}$. We denote the component in the plane α by $\boldsymbol{x}_\alpha$ (see figure 5). Then $\boldsymbol{x}$ may be written

$$\boldsymbol{x} = \boldsymbol{x}_\alpha + \lambda \boldsymbol{n}, \quad \text{with} \quad \boldsymbol{n}^{\mathrm{T}} g \boldsymbol{x}_\alpha = 0, \tag{B 15}$$

for some scalar λ.

The vector $\boldsymbol{x}_\alpha$ will be called the 'g-projection' of the vector $\boldsymbol{x}$ onto the plane α.

PROPERTY 2. *If* $\boldsymbol{P}$ *is a linear bivector,* $\boldsymbol{P} = \mu \boldsymbol{s}$ *(say), then* $\boldsymbol{Q}$ *is any bivector in the conjugate plane to* $\boldsymbol{s}$ *with respect to the g-metric ellipsoid.*

Proof. In this case, (B 1) reduces to

$$\boldsymbol{s}^{\mathrm{T}} g \boldsymbol{Q}^+ = \boldsymbol{s}^{\mathrm{T}} g \boldsymbol{Q}^- = 0, \tag{B 16}$$

which expresses the fact that $\boldsymbol{Q}^+$ and $\boldsymbol{Q}^-$ are in the plane conjugate to $\boldsymbol{s}$ with respect to the metric ellipsoid.

PROPERTY 3. *If neither* $\boldsymbol{P}$, *nor* $\boldsymbol{Q}$ *is a linear bivector, the plane of the ellipse of* $\boldsymbol{Q}$ *may not contain the conjugate direction to the plane of the ellipse of* $\boldsymbol{P}$ *with respect to the metric ellipsoid.*

Proof. Let $\boldsymbol{n} \neq 0$ be a vector in the conjugate direction to the plane α of $\boldsymbol{P}^+$, $\boldsymbol{P}^-$ with respect to the metric ellipsoid:

$$\boldsymbol{P}^{+\mathrm{T}} g \boldsymbol{n} = \boldsymbol{P}^{-\mathrm{T}} g \boldsymbol{n} = 0. \tag{B 17}$$

We now show that $\boldsymbol{n}$ may not lie in the plane of $\boldsymbol{Q}^+$ and $\boldsymbol{Q}^-$. For, suppose that $\boldsymbol{n}$ lies in this plane. Then, for some real scalars ν and γ $(\nu^2 + \gamma^2 \neq 0)$,

$$\boldsymbol{n} = \nu \boldsymbol{Q}^+ + \gamma \boldsymbol{Q}^-, \tag{B 18}$$

and (B 17) then reads

$$\nu \boldsymbol{P}^{+\mathrm{T}} g \boldsymbol{Q}^+ + \gamma \boldsymbol{P}^{+\mathrm{T}} g \boldsymbol{Q}^- = \nu \boldsymbol{P}^{-\mathrm{T}} g \boldsymbol{Q}^+ + \gamma \boldsymbol{P}^{-\mathrm{T}} g \boldsymbol{Q}^- = 0. \tag{B 19}$$

Eliminating ν and γ from these equations gives

$$(\boldsymbol{P}^{+\mathrm{T}} g \boldsymbol{Q}^+)(\boldsymbol{P}^{-\mathrm{T}} g \boldsymbol{Q}^-) = (\boldsymbol{P}^{+\mathrm{T}} g \boldsymbol{Q}^-)(\boldsymbol{P}^{-\mathrm{T}} g \boldsymbol{Q}^+), \tag{B 20}$$

and then using (B 2) and (B 3) we obtain

$$(\boldsymbol{P}^{+\mathrm{T}} g \boldsymbol{Q}^{+})^2 + (\boldsymbol{P}^{+\mathrm{T}} g \boldsymbol{Q}^{-})^2 = (\boldsymbol{P}^{-\mathrm{T}} g \boldsymbol{Q}^{-})^2 + (\boldsymbol{P}^{-\mathrm{T}} g \boldsymbol{Q}^{+})^2 = 0. \tag{B 21}$$

Now, from (B 21)

$$\boldsymbol{P}^{+\mathrm{T}} g \boldsymbol{Q}^{+} = \boldsymbol{P}^{-\mathrm{T}} g \boldsymbol{Q}^{+} = 0, \tag{B 22}$$

which means that $\boldsymbol{Q}^+$ is in the direction conjugate to the plane of $\boldsymbol{P}^+$ and $\boldsymbol{P}^-$ with respect to the metric ellipsoid. Again from (B 21),

$$\boldsymbol{P}^{+\mathrm{T}} g \boldsymbol{Q}^{-} = \boldsymbol{P}^{-\mathrm{T}} g \boldsymbol{Q}^{-} = 0, \tag{B 23}$$

which means that $\boldsymbol{Q}^-$ is also in the direction conjugate to the plane of $\boldsymbol{P}^+$ and $\boldsymbol{P}^-$ with respect to the metric ellipsoid. This means that $\boldsymbol{Q}^+$ and $\boldsymbol{Q}^-$ are parallel contrary to hypothesis.

Remark 3. In the special case when $g = g1$, the g-metric ellipsoid is a sphere. The direction conjugate to any plane with respect to the sphere is the normal to that plane. In this case equation (B 1) reads $\boldsymbol{P}\cdot\boldsymbol{Q} = 0$ and property 3 is a statement that the planes of $\boldsymbol{P}$ and of $\boldsymbol{Q}$ may not be orthogonal (see Hayes 1984).

PROPERTY 4. *If neither* $\boldsymbol{P}$ *nor* $\boldsymbol{Q}$ *is a linear bivector, the g-projection of the ellipse of* $\boldsymbol{Q}$ *onto the plane of the ellipse of* $\boldsymbol{P}$ *is similar, and similarly situated, to the polar reciprocal of the ellipse of* $\boldsymbol{P}$ *with respect to the section of the g-metric ellipsoid by the plane of the bivector* $\boldsymbol{P}$.

Proof. Let $\boldsymbol{n} \neq 0$ be a vector in the conjugate direction to the plane α of $\boldsymbol{P}^+$ and $\boldsymbol{P}^-$ with respect to the metric ellipsoid, so that (B 17) holds, or equivalently,

$$\boldsymbol{P}^{\mathrm{T}} g \boldsymbol{n} = 0. \tag{B 24}$$

The vectors $\boldsymbol{Q}^+$, $\boldsymbol{Q}^-$ may now be decomposed as in (B 15):

$$\boldsymbol{Q}^+ = \boldsymbol{Q}^+_\alpha + \lambda^+ \boldsymbol{n}, \quad \text{with} \quad \boldsymbol{n}^{\mathrm{T}} g \boldsymbol{Q}^+_\alpha = 0, \tag{B 25}$$

$$\boldsymbol{Q}^- = \boldsymbol{Q}^-_\alpha + \lambda^- \boldsymbol{n}, \quad \text{with} \quad \boldsymbol{n}^{\mathrm{T}} g \boldsymbol{Q}^-_\alpha = 0, \tag{B 26}$$

for some scalars λ^+ and λ^-, and where $\boldsymbol{Q}^+_\alpha$, $\boldsymbol{Q}^-_\alpha$ are the g-projections of $\boldsymbol{Q}^+$, $\boldsymbol{Q}^-$ onto the plane α of the bivector $\boldsymbol{P}$. In terms of bivectors, (B 25) and (B 26) may be written

$$\boldsymbol{Q} = \boldsymbol{Q}_\alpha + \lambda \boldsymbol{n}, \quad \text{with} \quad \boldsymbol{n}^{\mathrm{T}} g \boldsymbol{Q}_\alpha = 0, \tag{B 27}$$

with $\boldsymbol{Q}_\alpha = \boldsymbol{Q}^+_\alpha + \mathrm{i}\boldsymbol{Q}^-_\alpha$ and $\lambda = \lambda^+ + \mathrm{i}\lambda^-$. Introducing (B 27) into (B 1) and using (B 24), we note that (B 1) is equivalent to

$$\boldsymbol{P}^{\mathrm{T}} g \boldsymbol{Q}_\alpha = 0. \tag{B 28}$$

The bivectors $\boldsymbol{P}$ and $\boldsymbol{Q}_\alpha$ entering (B 28) have their ellipses in the same plane α, so that they obey property 1 where the metric ellipse is the elliptical section of the g-metric ellipsoid by the common plane α. Thus the ellipse of $\boldsymbol{Q}_\alpha$ is similar, and similarly situated, to the polar reciprocal of the ellipse of $\boldsymbol{P}$ with respect to the elliptical section of the metric ellipsoid by the plane α. This completes the proof because the ellipse of $\boldsymbol{Q}_\alpha$ is the g-projection of the ellipse of $\boldsymbol{Q}$ onto the plane α.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Appendix C. Isotropic eigenbivectors of a complex symmetric matrix with respect to a metric

Let X be a complex symmetric 3×3 matrix, and g a real symmetric positive definite 3×3 matrix. The eigenbivectors $\boldsymbol{P}$ and the eigenvalues λ of the matrix X with respect to the metric g are the solutions of the eigenvalue problem

$$X\boldsymbol{P} = \lambda g \boldsymbol{P}. \tag{C 1}$$

A bivector $\boldsymbol{P}$ will be said to be an isotropic eigenbivector of the matrix X with respect to the metric g when it is a solution of (C 1) satisfying

$$\boldsymbol{P}^{\mathrm{T}} g \boldsymbol{P} = 0. \tag{C 2}$$

A geometrical interpretation of (C 2) is given in Appendix B (Remark 2).

Here it is shown that a necessary and sufficient condition that the complex symmetric matrix X have an isotropic eigenbivector with respect to the metric g is that this matrix have a double or triple eigenvalue with respect to the metric g.

We first assume that $\boldsymbol{P}$ is an isotropic eigenbivector of X with respect to the metric g and show that the corresponding eigenvalue λ is at least double. Next we assume that λ is a double eigenvalue of the eigenvalue problem (C 1) and show that corresponding to λ there is either a simple or a double infinity of eigenbivectors $\boldsymbol{P}$. In the case of a simple infinity they are all isotropic with respect to the metric g; in the case of a double infinity there are two non parallel isotropic eigenbivectors with respect to this metric, each defined up to an arbitrary scalar factor. Finally we assume that λ is a triple eigenvalue of the eigenvalue problem (C 1) and show that corresponding to λ there is either a simple or a double or a triple infinity of eigenbivectors $\boldsymbol{P}$. In the case of a simple infinity they are all isotropic with respect to the metric g; in the case of a double infinity there is one isotropic eigenbivector with respect to this metric, defined up to an arbitrary scalar factor; in the case of a triple infinity every isotropic bivector with respect to the metric g is an eigenbivector.

Property 1. *If $\boldsymbol{P}$ is an isotropic eigenbivector of the matrix X with respect to the metric g, then the corresponding eigenvalue λ is at least double.*

Proof. As $\boldsymbol{P}^{\mathrm{T}} g \bar{\boldsymbol{P}}$ is positive and because eigenbivectors may be multiplied by an arbitrary scalar factor, it may be assumed without loss of generality that

$$\boldsymbol{P}^{\mathrm{T}} g \bar{\boldsymbol{P}} = 1. \tag{C 3}$$

Let $\boldsymbol{n}$ be the real vector defined (up to a $\pm$ sign) by

$$\boldsymbol{P}^{+\mathrm{T}} g \boldsymbol{n} = \boldsymbol{P}^{-\mathrm{T}} g \boldsymbol{n} = 0, \quad \boldsymbol{n}^{\mathrm{T}} g \boldsymbol{n} = 1. \tag{C 4}$$

Then the matrix T defined by

$$T = (\boldsymbol{P} | \bar{\boldsymbol{P}} | \boldsymbol{n}) \tag{C 5}$$

is non singular because $\boldsymbol{P}$, $\bar{\boldsymbol{P}}$ and $\boldsymbol{n}$ are linearly independent. From (C 1)–(C 4) it follows that this matrix T transforms g and X into $\hat{g}$ and $\hat{X}$ given by

$$\hat{g} = T^{\mathrm{T}} g T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat{X} = T^{\mathrm{T}} X T = \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & \alpha & \beta \\ 0 & \beta & \mu \end{pmatrix}, \tag{C 6}$$

where $$\alpha = \bar{\boldsymbol{P}}^{\mathrm{T}} X \bar{\boldsymbol{P}}, \quad \mu = \boldsymbol{n}^{\mathrm{T}} X \boldsymbol{n}, \quad \beta = \boldsymbol{n}^{\mathrm{T}} X \bar{\boldsymbol{P}}. \tag{C 7}$$

The equation $\det(X - xg) = 0$, or equivalently $\det(\hat{X} - x\hat{g}) = 0$ for the determination of the eigenvalues x thus reads
$$(x-\lambda)^2 (x-\mu) = 0, \tag{C 8}$$
which shows that the root λ is at least double.

Remark 1. *If* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are any two bivectors, isotropic with respect to the metric* g, *and also satisfying*
$$\boldsymbol{B}^{\mathrm{T}} g \boldsymbol{A} = 0, \tag{C 9}$$
then $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are parallel.*

Proof. As g is positive definite symmetric it possesses the square root $g^{\frac{1}{2}}$. Then, writing
$$\boldsymbol{A}' = g^{\frac{1}{2}} \boldsymbol{A}, \quad \boldsymbol{B}' = g^{\frac{1}{2}} \boldsymbol{B}, \tag{C 10}$$
we have
$$\boldsymbol{A}' \cdot \boldsymbol{A}' = \boldsymbol{B}' \cdot \boldsymbol{B}' = \boldsymbol{B}' \cdot \boldsymbol{A}' = 0. \tag{C 11}$$

It then follows from the result of Synge (1966) that $\boldsymbol{A}'$ and $\boldsymbol{B}'$ are parallel: $\boldsymbol{A}' = \lambda \boldsymbol{B}'$ for some λ, and hence $\boldsymbol{A} = \lambda \boldsymbol{B}$.

Property 2. *If the matrix* X *has a double eigenvalue* λ *and a simple eigenvalue* $\mu \neq \lambda$ *with respect to the metric* g, *then, corresponding to the eigenvalue* λ, *either*

(a) X *has a simple infinity of eigenbivectors with respect to the metric* g, *all isotropic with respect to this metric, or*

(b) X *has a double infinity of eigenbivectors with respect to the metric* g; *among these there are two non-parallel eigenbivectors, isotropic with respect to this metric, defined up to an arbitrary scalar factor.*

Proof. Let $\boldsymbol{Q}$ be an eigenbivector corresponding to the simple eigenvalue μ. It is not isotropic with respect to the metric g for otherwise μ would be at least double (property 1). Without loss of generality, the bivector $\boldsymbol{Q}$ may be normalized with respect to the metric g:
$$X\boldsymbol{Q} = \mu g \boldsymbol{Q}, \quad \boldsymbol{Q}^{\mathrm{T}} g \boldsymbol{Q} = 1. \tag{C 12}$$

For the eigenbivectors corresponding to the double eigenvalue λ, two possibilities have to be considered: either they are all isotropic with respect to the metric g, or at least one of them is not isotropic with respect to this metric.

Possibility (a). Suppose that all the eigenbivectors corresponding to the double eigenvalue λ are isotropic with respect to the metric g. Let $\boldsymbol{P}$ and $\boldsymbol{P}'$ be any two of them:
$$X\boldsymbol{P} = \lambda g \boldsymbol{P}, \quad X\boldsymbol{P}' = \lambda g \boldsymbol{P}', \quad \boldsymbol{P}^{\mathrm{T}} g \boldsymbol{P} = \boldsymbol{P}'^{\mathrm{T}} g \boldsymbol{P}' = 0. \tag{C 13}$$

Then $\boldsymbol{P} + \boldsymbol{P}'$ is also an eigenbivector corresponding to the eigenvalue λ, and from the assumption of possibility (a) it has to be isotropic with respect to the metric g:
$$(\boldsymbol{P} + \boldsymbol{P}')^{\mathrm{T}} g (\boldsymbol{P} + \boldsymbol{P}') = 0, \tag{C 14}$$
and hence
$$\boldsymbol{P}^{\mathrm{T}} g \boldsymbol{P}' = 0. \tag{C 15}$$

Thus, $\boldsymbol{P}$ and $\boldsymbol{P}'$ are both isotropic with respect to the metric g and are orthogonal with respect to this metric. This implies that $\boldsymbol{P}$ and $\boldsymbol{P}'$ are parallel (remark 1). Thus, for possibility (a) all the eigenbivectors are parallel. Hence X has a simple infinity of eigenbivectors corresponding to the double eigenvalue λ, all isotropic with respect to the metric g.

Possibility (*b*). Suppose now that corresponding to the double eigenvalue λ there is an eigenbivector which is not isotropic with respect to the metric g. Let $\boldsymbol{P}$ be this eigenbivector. Without loss of generality, it may be normalized with respect to the metric g:

$$X\boldsymbol{P} = \lambda g\boldsymbol{P}, \quad \boldsymbol{P}^{\mathrm{T}}g\boldsymbol{P} = 1. \tag{C 16}$$

Because $\boldsymbol{P}$ and $\boldsymbol{Q}$ are both eigenbivectors of the matrix X with respect to the metric g corresponding to different eigenvalues, it is easily shown by the standard argument that they are orthogonal with respect to the metric g:

$$\boldsymbol{P}^{\mathrm{T}}g\boldsymbol{Q} = 0. \tag{C 17}$$

Now choose the bivector $\boldsymbol{R}$ such that $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ form an orthonormal triad with respect to the metric g:

$$\boldsymbol{P}^{\mathrm{T}}g\boldsymbol{R} = \boldsymbol{Q}^{\mathrm{T}}g\boldsymbol{R} = 0, \quad \boldsymbol{R}^{\mathrm{T}}g\boldsymbol{R} = 1. \tag{C 18}$$

Then the matrix T defined by $T = (\boldsymbol{P}|\boldsymbol{Q}|\boldsymbol{R})$ is such that

$$T^{\mathrm{T}}gT = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T^{\mathrm{T}}XT = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \boldsymbol{R}^{\mathrm{T}}X\boldsymbol{R} \end{pmatrix}. \tag{C 19}$$

But λ is a double eigenvalue of the matrix X with respect to the metric g. Therefore $\lambda = \boldsymbol{R}^{\mathrm{T}}X\boldsymbol{R}$. Hence $\boldsymbol{R}$ defined by (C 18) is an eigenbivector of the matrix X with respect to the metric g corresponding to the eigenvalue λ. Any linear combination $a\boldsymbol{P}+b\boldsymbol{R}$ of $\boldsymbol{P}$ and $\boldsymbol{R}$ with coefficients a and b is also an eigenbivector corresponding to the eigenvalue λ. Thus, for possibility (*b*), X has a double infinity of eigenbivectors corresponding to the eigenvalue λ. Among these the bivectors $\boldsymbol{P}+\mathrm{i}\boldsymbol{R}$, $\boldsymbol{P}-\mathrm{i}\boldsymbol{R}$ and their scalar multiples are isotropic with respect to the metric g.

To complete the proof we note that both possibilities (*a*) and (*b*) are allowed. Indeed take for example

$$g = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{C 20}$$

Let $\boldsymbol{P}$ and $\boldsymbol{n}$ be given by

$$\boldsymbol{P}^{\mathrm{T}} = 2^{-\frac{1}{2}}(1, -1+\mathrm{i}, 0), \quad \boldsymbol{n}^{\mathrm{T}} = (0, 0, 1). \tag{C 21}$$

The bivector $\boldsymbol{P}$ is isotropic with respect to the metric g, and $\boldsymbol{P}$ and $\boldsymbol{n}$ satisfy (C 3) and (C 4). Choose any complex numbers λ, μ, α, β (with $\mu \neq \lambda$), and consider the matrix X defined by (C 6) with T given by (C 5):

$$X = \begin{pmatrix} 2\lambda+\mathrm{i}\alpha & \lambda-\frac{1}{2}(1-\mathrm{i})\alpha & 2^{-\frac{1}{2}}(1+\mathrm{i})\beta \\ \lambda-\frac{1}{2}(1-i)\alpha & \lambda-\frac{1}{2}\alpha & \mathrm{i}2^{-\frac{1}{2}}\beta \\ 2^{-\frac{1}{2}}(1+\mathrm{i})\beta & \mathrm{i}2^{-\frac{1}{2}}\beta & \mu \end{pmatrix}. \tag{C 22}$$

This matrix X has the double eigenvalue λ and the simple eigenvalue μ with respect to the metric g. Then $\beta^2 \neq \alpha(\mu-\lambda)$ leads to possibility (*a*) and $\beta^2 = \alpha(\mu-\lambda)$ leads to possibility (*b*).

Remark 2. Among all the linear combinations $a\boldsymbol{P}+b\boldsymbol{R}$ of two non parallel bivectors $\boldsymbol{P}$, $\boldsymbol{R}$

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there is always a linear bivector, that is a real vector up to a possibility complex scalar factor. Indeed, $a\boldsymbol{P}+b\boldsymbol{R}$ is a real vector when the coefficients $a = a^{+}+\mathrm{i}a^{-}$, $b = b^{+}+\mathrm{i}b^{-}$ are such that

$$(a\boldsymbol{P}+b\boldsymbol{R})^{-} = a^{+}\boldsymbol{P}^{-}+a^{-}\boldsymbol{P}^{+}+b^{+}\boldsymbol{R}^{-}+b^{-}\boldsymbol{R}^{+} = 0. \tag{C 23}$$

As $\boldsymbol{P}^{+}$, $\boldsymbol{P}^{-}$, $\boldsymbol{R}^{+}$, $\boldsymbol{R}^{-}$ are four vectors in a three-dimensional space they are linearly dependent, and thus equation (C 23) has a non-trivial solution for a^{+}, a^{-}, b^{+}, b^{-}.

PROPERTY 3. *If the matrix X has a triple eigenvalue λ with respect to the metric g, then, corresponding to the eigenvalue λ, either*

(*a*) *X has a simple infinity of eigenbivectors with respect to the metric g, all isotropic with respect to this metric, or*

(*b*) *X has a double infinity of eigenbivectors with respect to the metric g; among these there is one eigenbivector isotropic with respect to this metric, defined up to an arbitrary scalar factor, or*

(*c*) *X has a triple infinity of eigenbivectors with respect to the metric g; thus any bivector (in particular isotropic with respect to the metric g) is an eigenbivector of the matrix X with respect to this metric.*

Proof. For the eigenbivectors corresponding to the triple eigenvalue λ, two possibilities have to be considered: either they are all isotropic with respect to the metric g, or at least one of them is not isotropic with respect to this metric.

Possibility (*a*). Suppose that all the eigenbivectors corresponding to the triple eigenvalue λ are isotropic with respect to the metric g. Then using the same argument as in the proof of property 2 (possibility (*a*)) we note that all the eigenbivectors are parallel. Hence X has a simple infinity of eigenbivectors corresponding to the triple eigenvalue λ, all isotropic with respect to the metric g.

Possibility (*b*, *c*). Suppose now that corresponding to the triple eigenvalue λ there is an eigenbivector which is not isotropic with respect to the metric g. Let $\boldsymbol{P}$ be this eigenbivector. Without loss of generality, it may be normalized with respect to the metric g:

$$X\boldsymbol{P} = \lambda g\boldsymbol{P}, \quad \boldsymbol{P}^{\mathrm{T}}g\boldsymbol{P} = 1. \tag{C 24}$$

Now choose the bivectors $\boldsymbol{Q}$ and $\boldsymbol{R}$ such that $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ form an orthonormal triad with respect to the metric g:

$$\boldsymbol{P}^{\mathrm{T}}g\boldsymbol{R} = \boldsymbol{P}^{\mathrm{T}}g\boldsymbol{Q} = \boldsymbol{Q}^{\mathrm{T}}g\boldsymbol{R} = 0, \quad \boldsymbol{Q}^{\mathrm{T}}g\boldsymbol{Q} = \boldsymbol{R}^{\mathrm{T}}g\boldsymbol{R} = 1. \tag{C 25}$$

Then the matrix T defined by $T = (\boldsymbol{P}|\boldsymbol{Q}|\boldsymbol{R})$ is such that

$$T^{\mathrm{T}}gT = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T^{\mathrm{T}}XT = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \gamma & \delta \\ 0 & \delta & \nu \end{pmatrix}. \tag{C 26}$$

with

$$\gamma = \boldsymbol{Q}^{\mathrm{T}}X\boldsymbol{Q}, \quad \nu = \boldsymbol{R}^{\mathrm{T}}X\boldsymbol{R}, \quad \delta = \boldsymbol{Q}^{\mathrm{T}}X\boldsymbol{R}. \tag{C 27}$$

But λ is a triple eigenvalue of the matrix X with respect to the metric g. Therefore,

$$\gamma = \lambda+\mathrm{i}\delta, \quad \nu = \lambda-\mathrm{i}\delta. \tag{C 28}$$

Subcase (*b*) : $\delta \neq 0$. For $\delta \neq 0$, the eigenbivectors of the matrix $T^{\mathrm{T}}XT$ with respect to the unit matrix are $(1,0,0)$, $(0,\mathrm{i},1)$ and any linear combination of these two vectors. Thus the eigenbivectors of the matrix X with respect to the metric g are $\boldsymbol{P}$, $\mathrm{i}\boldsymbol{Q}+\boldsymbol{R}$ and any linear combination $a\boldsymbol{P}+b(\mathrm{i}\boldsymbol{Q}+\boldsymbol{R})$ of these bivectors with coefficients a and b. Thus X has a double infinity of eigenbivectors corresponding to the triple eigenvalue λ. Among these only the bivector $\mathrm{i}\boldsymbol{Q}+\boldsymbol{R}$ and its scalar multiples are isotropic with respect to the metric g.

Subcase (*c*): $\delta = 0$. For $\delta = 0$, one has $T^{\mathrm{T}}XT = \lambda 1$ and thus $X = \lambda g$. Then any bivector (and in particular any isotropic bivector with respect to the metric g) is an eigenbivector of the matrix X with respect to the metric g.

To complete the proof we note that the three possibilities (*a*), (*b*), (*c*) are allowed. Indeed, as in the proof of property 2, take g and X respectively given by (C 20) and (C 22), but now with $\mu = \lambda$. Thus the matrix X has now the triple eigenvalue λ with respect to the metric g. Then $\beta \neq 0$ leads to possibility (*a*), $\beta = 0$ and $\alpha \neq 0$ leads to possibility (*b*), and $\beta = \alpha = 0$ leads to possibility (*c*).

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